STUDY OF QUANTUM RESOURCES TO DEVISE ADVANCED COMMUNICATION PROTOCOLS

Thesis submitted for the partial fulfillment of the requirements for the degree Doctor of Philosophy in Science

by

Ram Krishna Patra

Department of Physics
Faculty of Natural and Mathematical Sciences
Presidency University
Kolkata, India

2025

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Under the Supervision of

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I dedicate my thesis to my caring and supportive parents, Rama Patra and Satya Narayan Patra.

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Declaration

I hereby declare that this thesis contains original research work carried out by me under the guidance of Dr. Manik Banik, Associate Professor, Department of Physics of Complex Systems, S.N. Bose National Centre for Basic Sciences (SNBNCBS), Kolkata, India as part of the PhD programme.

All information in this document have been obtained and presented in accordance with academic rules and ethical conduct.

I also declare that, as required by these rules and conduct, I have fully cited and referenced all materials and results that are not original to this work.

I also declare that, this work has not been submitted for any degree either in part or in full to any other institute or University before.

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CERTIFICATE

This is to certify that the thesis entitled "STUDY OF QUANTUM RESOURCES TO DEVISE ADVANCED COMMUNICATION PROTOCOLS" submitted by **Shri Ram Krishna Patra**, Registration Number RC001-22RS209120347 and date of registration 12th August 2022, in partial fulfilment of the requirements for the award of "Doctor of Philosophy", is a record of bona-fide research work carried out by him under my supervision.

Neither his thesis nor any part of the thesis has been submitted for any degree/diploma or any other academic award anywhere before.

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Abstract

Quantum information processing has emerged as an enriched research field, both from theoretical insights and experimental observations, and fueled by its vast practical applications. This field leverages fundamental principles of quantum mechanics, such as superposition, entanglement, and measurement incompatibility to enable novel communication, computation, and cryptography approaches. Exploiting these uniquely quantum phenomena it enables information processing tasks with efficiencies and security levels unattainable within classical paradigms. This thesis is devoted to the study of various quantum resources and their application in communication tasks. Firstly, we analyze the role of shared correlation in enhancing the utility of classical channels in the most basic communication scenario. With this aim, we propose a class of two-party communication games and show that the games cannot be won given a noiseless 1-bit classical channel from the sender to the receiver without any assistance of shared correlation. Interestingly, the goal can be perfectly achieved if the channel is assisted with classical shared randomness. This resembles a classical advantage similar to the quantum superdense coding phenomenon where pre-shared entanglement can enhance the communication utility of a perfect quantum communication line. Quite surprisingly, we show that a qubit communication without any assistance of classical shared randomness can achieve the goal and hence establish a unique quantum advantage in the simplest communication scenario. In pursuit of a deeper origin of this advantage, we show that an advantageous quantum strategy must invoke quantum interference both at the sender's encoding step and at the receiver's decoding step. We also study the communication utility of a class of non-classical toy systems described by symmetric polygonal state spaces. We come up with communication tasks that can be achieved neither with 1-bit of classical communication nor by communicating a polygon system, whereas 1-qubit communication yields a perfect strategy, establishing quantum advantage over them. To this end, we show that the quantum advantages are robust against imperfect encodings-decodings, making the protocols implementable with presently available quantum technologies. Thereafter, we move on to the communication complexity scenario in a prepare and measure setup. We have considered the task of Random Access Code (RAC), where the receiver aims to guess different bits of the sender's input string based on the received input. Here, we have demonstrated that two extremal compositions of local quantum systems, known as maximal composition and minimal composition, are more advantageous than standard quantum composition. Finally, to harness these advantages in practical applications, we consider the certification of these resources benificial in various communication tasks. Specifically, we have considered the semi-device-independent certification of states in the maximal composition of local quantum systems, referred to as Beyond Quantum States (BQS) in generalized quantum input Bell scenario. Additionally, we have also explored the foundational implications of these communication advantages. We have also shown that both the extremal compositions of local quantum systems are not compatible with the information causality principle.

Ramkrishna Patra
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List of publications

Publications and communicated works relevant to the thesis:

- "*Classical analogue of quantum superdense coding and communication advantage of a single quantum system*": **Ram Krishna Patra**, S. G. Naik, E. P. Lobo, S. Sen, T. Guha, S. S. Bhattacharya, M. Alimuddin, M. Banik; Quantum **8**, 1315 (2024).

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- "*Local-Data-Hiding and Causal Inseparability: Probing Indefinite Causal Structures with Cryptographic Primitives*": Sahil Gopalkrishna Naik, Samrat Sen, **Ram Krishna Patra**, Ananya Chakraborty, Mir Alimuddin, Manik Banik, Pratik Ghosal; arXiv:2407.20543 [quant-ph].
- "*Nonlocal Locking of Observable Quantities: A Faithful Signature of Nonclassical Correlations*": Mir Alimuddin, Snehasish Roy Chowdhury, **Ram Krishna Patra**, Subhendu B. Ghosh, Tommaso Tufarelli, Gerardo Adesso, Manik Banik; arXiv:2407.08292 [quant-ph].
- "*Scalable & Noise-Robust Communication Advantage of Multipartite Quantum Entanglement*": Ananya Chakraborty, **Ram Krishna Patra**, Kunika Agarwal, Samrat Sen, Pratik Ghosal, Sahil Gopalkrishna Naik, Manik Banik; arXiv:2409.13223 [quant-ph].

- "*Nonlocality-Assisted Enhancement of Error-Free Communication in Noisy Classical Channels*": Kunika Agarwal, Sahil Gopalkrishna Naik, Ananya Chakraborty, Samrat Sen, Pratik Ghosal, Biswajit Paul, Manik Banik, **Ram Krishna Patra**; arXiv:2412.04779 [quant-ph].

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Chapter 1

General Introduction

1.1 Motivation

The Information Age, which began in the mid-20th century, had a profound impact on our modern civilization. It has brought about an efficient and economical shift in traditional industries that were developed during the 17th century's Industrial Revolution, primarily based upon information technology. Claude Shannon laid the groundwork for information theory in his seminal work "A Mathematical Theory of Communication" in 1948 [1]. His work has formed the basis for all present-day communication tasks. Before Shannon, Harry Nyquist [2] and Ralph Hartley [3] also made significant contributions to the field. Shannon was the first to quantify the amount of information in terms of "bits" by defining entropy as a measure of uncertainty of a random variable. The operational meaning of this entropy was provided by the source coding theorem, which sets the limit on the optimal data compression of Independent Identically Distributed (IID) sources [1]. In the noisy channel coding theorem, he also set limits on the rate of reliable information transmission using noisy channels [1]. However, in all his work, the physical systems and channels are considered to be classical.

John Von Neumann introduced the concept of entropy for quantum systems in 1932 [4], predating Shannon's work. Quantum entropy is an extension of Gibbs entropy from classical statistical mechanics to quantum statistical mechanics. Benjamin Schumacher first established the limit on compressing information of quantum systems (QI) in a smaller number of systems [5]. This Schumacher compression theorem, a quantum counterpart of Shannon's Source coding theorem, laid the foundation of quantum information theory. Over time, as demand increased, all classical information processing and communication tasks have been transformed into their quantum information processing counterparts. It has been demonstrated that quantum resources are more advantageous than classical resources in various information-processing tasks. Moreover, the existence of communication tasks that are achievable in quantum theory but have no analog in classical theory, was discovered, such as Quantum Key Distribution [6], Super Dense Coding [7], Quantum Teleportation [8], and Random number generation [9], certification, and amplification [10, 11]. These advantages are driven by quantum superposition, coherence, steering, and entanglement. The concept of entanglement arises from the tensor product structure of quantum mechanics, where the composite system cannot be separated into individual subsystems. The term "Entanglement" was first coined by Schrödinger to explain spatially separated correlated systems [12]. Later, Einstein, Podolsky, and Rosen have shown that the existence of entanglement leads to the incompleteness of quantum mechanics using the Local-realism hypothesis, known as the EPR paradox [13]. The term "Steering" was also introduced by Schrödinger to describe the nonclassical feature of quantum entanglement, where the measurement choice on one particle can affect the state of another spatially separated particle [12].

In 1964, John Bell provided a mathematical formulation to the philosophical debate of the EPR paradox by introducing mathematical inequalities known as Bell inequalities [14, 15]. These inequalities are satisfied by any local-realistic theory. It has been demonstrated that quantum mechanics violates this inequality, indicating that quantum mechanics cannot be explained by local hidden variable models, meaning that quantum mechanics is "nonlocal". Several experiments have shown the violation of Bell inequality, pointing to the nonlocality of nature [16–18]. All Nonlocal correlations have been shown to offer advantages in communication complexity tasks [19, 20]. However, understanding the role of entanglement in communication complexity is challenging, as it does not straightforwardly follow from nonlocality due to their complex relationship. Werner has shown that entanglement is a necessary criterion for nonlocality, but it is not sufficient [21]. The hierarchy among Bell nonlocality, EPR steering, and entanglement has also been explored, revealing a strict ordering among them, with EPR steering lying between the other two [22].

Though entanglement and nonlocal correlation are advantageous in various communication tasks, the classical capacity of a quantum system without any assistance of entanglement and nonlocal correlation is bounded. The bound on the rate of classical information that can be transmitted using a quantum system was given by the Holevo theorem [23] later generalized by Frenkel Weiner [24], which says that any utility of N level quantum system is same as N level classical system. Even with this no-go theorem, assistance quantum entanglement and nonlocality lead to tremendous advantages in quantum communication and quantum information processing. In 1970, Bennett *et. al.* showed that assistance of no signaling quantum entanglement can double the classical capacity of the quantum bit [7]. The bound of the Holevo theorem is applicable in the basic communication scenario where the sender (Alice) is receiving a classical random variable $x \in \mathscr{X}$ and Bob outputs an m bit string $b \in \mathscr{B}$ and according to the correlation $P(b|x)$ generated by Alice and Bob referee gives them some payoff $\beta : P(\mathscr{B}|\mathscr{X}) \to \mathbb{R}$. In this scenario, a qubit is no longer advantageous over cbit, considering Shared Randomness as a Free resource [23, 24]. But considering Shared Randomness (SR) as free resources, one loses the importance and utility of Shared Randomness in different communication tasks. This thesis aims to discover the advantages of shared randomness in different communication tasks by considering SR as a costly resource.

In another communication scenario, both Alice and Bob receive classical random variables $x \in \mathscr{X}$ and $y \in \mathscr{Y}$, and Bob aims to calculate a function of the input bit, denoted as $f(x,y)$. Quantum resources have been shown to be more advantageous than classical resources in this Communication Complexity scenario, as demonstrated in Quantum Random Access Code (QRAC) [25, 26], GHZ game [27], and other examples. It's interesting to explore the source of these advantages, which leads to the study of Generalized Probability Theory (GPT) [28] and its role in communication complexity [29]. Studies have shown that Box World, a class of GPT, can provide an unbounded advantage by simplifying any communication complexity problem [19]. In this thesis, we aim to explore the benefits of different compositions of local quantum systems, including Beyond Quantum Systems (BQS) in communication complexity scenarios.

To utilize these communication advantages in real life we should be able to certify the resources. Device-independent and semi-device-independent certification of quantum entanglement and nonlocality is a well-established field [30, 31]. We also aim to certify different advantageous resources to harness their utility in real-life applications. Communication tasks not only help us to reveal the supremacy of quantum systems over classical systems, but they also give us a deeper understanding of nature, which leads to different physical principles of nature *i.e.* Nontrivial Communication complexity [19], Information Causality

[32]. We are also exploring the foundational implications of communication advantages of Beyond Quantum Systems (BQS).

1.2 Postulates of Quantum Theory

In the early 20th century, phenomena such as black-body radiation, the photoelectric effect, and Compton scattering, among others, could not be adequately explained using the classical physics theories of the time. To account for these experimental results in a consistent manner, a new theoretical framework—quantum mechanics (QM)—was developed. Quantum mechanics, as an operational theory, is built around three core principles: 1) the preparation of a quantum state, 2) the transformation of that state, and 3) the measurement of the state. Like other operational theories, QM is defined by a set of foundational postulates that govern these processes.

Postulate 1 : State

Every quantum mechanical system S is associated with a separable complex Hilbert space $\mathscr{H}_S$. The system is completely described by its 'density operator', which is a positive operator ρ with trace one acting on the system's state space.

The simplest quantum mechanical system is of Hilbert dimension two - called a qubit. A qubit can be the polarization degree of freedom of a single photon or spin degree of freedom of a spin-$\frac{1}{2}$ particle. The set of all positive, trace one, self-adjoint linear operators $\mathscr{D}(\mathscr{H}_S)$ acting on $\mathscr{H}_S$ is the state space for this system. Therefore the state space consists of elements ρ which satisfies the following conditions, (i) $\rho \geq 0$, (ii) $\text{Tr}(\rho) = 1$ and (iii) $\rho = \rho^\dagger$. Such a set is a convex-compact subset of the space of linear operators acting on $\mathscr{H}_S$. The set of density operators can be classified into two types : Pure states and Mixed states.

Pure state: A state $\rho \in \mathscr{D}(\mathscr{H}_S)$ is said to be a pure state if it cannot be decomposed as a convex mixture of other states in $\mathscr{D}(\mathscr{H}_S)$. The one-rank projection operators $|\psi\rangle\langle\psi| \in \mathscr{D}(\mathscr{H}_S)$ correspond to pure states, where $|\psi\rangle$ is a vector of unit norm in $\mathscr{H}_S$. Therefore, pure states are often denoted as a unit vector $|\psi\rangle \in \mathscr{H}_S$. Pure state ρ also satisfies the criteria $\rho^2 = \rho$. This implies $\text{Tr}(\rho^2) = 1$ if ρ corresponds to a pure state.

Mixed state: Mixed states are nothing but a classical statistical mixture of pure states. If a physical system is in a classical mixture of pure state $|\psi_i\rangle$ with probability p_i, then it can be written as $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. Unlike pure state $\text{Tr}(\rho^2) < 1$ for mixed states. It is important to note that a particular mixed state does not have a unique decomposition in terms of pure states, it can have infinitely many decompositions. For example a pure state ρ can be decomposed as $\rho = \sum_m p_m |\psi_m\rangle\langle\psi_m| = \sum_n q_n |\phi_n\rangle\langle\phi_n|$ where $\{p_m\}$ and $\{q_n\}$ are probabilities *i.e.*, $p_m \geq 0, q_n \geq 0$ and $\sum_m p_m = \sum_n q_n = 1$.

Postulate 2 : Measurement

Quantum measurements are represented by a set of operators M_k acting on the system's Hilbert space, which satisfies the completeness condition: $\sum_k M_k^\dagger M_k = \mathbb{I}$. The probability that k^{th} outcome will occur if a measurement M associated with the collection of operators $\{M_k\}$ is performed on a system prepared in some state ρ, is given by

$$p(k\,|\rho, M) = Tr(M_k^\dagger M_k\, \rho)$$

and the post-measurement state conditioned that the k^{th} outcome has occurred, is given by

$$\rho_k = \frac{M_k \rho M_k^\dagger}{Tr(M_k^\dagger M_k \rho)}$$

Projective measurement: Projective measurement, P (von Neumann measurement), is a special class of quantum measurement consisting of a set of mutually orthogonal projectors satisfying the completeness relation, $P \equiv \{P_m | P_m P_{m'} = \delta_{m,m'} P_m,\ \sum_m P_m = \mathbb{I}\}$. For example, a projective operator can be described as a Hermitian operator M having spectral decomposition as $M = \sum_m m P_m$ where P_m is the orthogonal projector of observable M associated with the eigenvalue m. The average value of the observable M for a given state ρ is given by $\langle M \rangle_\rho = \sum_m m P(m|\rho) = \mathrm{Tr}(M\rho)$.

POVM: The 'positive-operator-valued-measure', abbreviated as POVM, represents the more general kind of quantum measurements that can be performed on a system. A POVM is defined as a collection of positive operators $\{E_i\}$, where $\sum_i E_i = \mathbb{I}$. The POVM elements E_i are often referred to as effects. If a POVM $\{E_i\}$ is measured on a system prepared in state ρ, the probability of clicking the i^{th} outcome is given by $p(i) = \mathrm{Tr}[\rho E_i]$. To fit the POVM formalism in the general measurement framework, one can simply consider a collection of measurement operators $\{M_i\}$ as $M_i = U_i\sqrt{E_i}$, such that $E_i = M_i^\dagger M_i$. However, for a particular POVM measurement, the choice of measurement operators $\{M_i\}$ is not unique [33]. Importantly, every POVM can be visualized as a projective measurement in a larger dimensional Hilbert space.

Postulate 3 : Dynamics

The time evolution of a closed quantum system is described by a unitary transformation. If the state of the system evolves from $\rho(t_0)$ at time t_0 to the state $\rho(t_1)$ at time $t_1(> t_0)$, then $\rho(t_1)$ is related to $\rho(t_0)$ by a unitary operator $U(t_0,t_1)$ which depends only on the times t_0 and t_1, i.e.,

$$\rho(t_1) = U(t_0,t_1)\rho(t_0)U(t_0,t_1)^\dagger$$

This discrete-time description of quantum state evolution using unitary operators can be obtained from the continuous time description of the Schrödinger equation.

$$i\hbar\frac{d|\psi\rangle}{dt} = H|\psi\rangle$$

Here H is the Hamiltonian operator associated with the closed system and $\hbar$ is the *Planck's constant*. From the above equation, the unitary operation corresponding to the evolution of a quantum state from $|\psi_0\rangle$ at time t_0 to the state $|\psi_1\rangle$ at time t_1 is given by

$$U(t_0,t_1) = \exp[\frac{-iH(t_1-t_0)}{\hbar}].$$

Postulate 4 : Composite System

The Hilbert space associated with a composite quantum system is the tensor product of the Hilbert spaces associated with its component quantum systems. i.e.,

$$\mathcal{H}_{1,2,\cdots,n} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_n$$

An n-partite composite quantum state is a density operator $\rho_{1,2,\cdots,n}$ belonging in $\mathcal{D}(\mathcal{H}_{1,2,\cdots,n})$, where $\mathcal{H}_{1,2,\cdots,n} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_n = \bigotimes_{i=1}^n \mathcal{H}_i$. If a composite state $\rho_{1,2,\cdots,n} \in \mathcal{D}(\bigotimes_{i=1}^n \mathcal{H}_i)$ is of the form $\rho_{1,2,\cdots,n} = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_n$, with $\rho_i \in \mathcal{D}(\mathcal{H}_i)$, then the state is called a product state. Convex combinations of such product states are called separable states *i.e.*

$$\rho^{Sep}_{A_1,\cdots,A_n} = \sum_\lambda p_\lambda \rho^\lambda_{A_1} \otimes \cdots \otimes \rho^\lambda_{A_N} \quad (1.1)$$

If $sep(\mathcal{H}_A \otimes \mathcal{H}_B)$ denotes the set of separable states, composite states that do not belong to $sep(\mathcal{H}_A \otimes \mathcal{H}_B)$ are called entangled states. That is, if $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$

is entangled, then it cannot be expressed as a convex combination of product states *i.e.* in the form of 1.1.

Necessary and sufficient criteria for detecting entanglement in a qubit-qubit system ($\mathbb{C}^2 \otimes \mathbb{C}^2$) and qubit-qutrit system ($\mathbb{C}^2 \otimes \mathbb{C}^3$) is given bt Peres and Horodechki Positive Partial Transpose (PPT) criteria [34, 35]. This condition states that the state ρ_{AB} is entangled if and only if $\rho_{AB}^{T_A}$ *i.e.* the partial transpose of ρ_{AB} has at least one negative eigenvalue.

1.3 Qubit : Elementary Quantum System

The simplest and basic example of quantum systems corresponds to a spin $1/2$ particle, also known as a qubit. The Hilbert space associated with a qubit is $\mathbb{C}^2$ *i.e.* two-dimensional complex Hilbert space. The state space of a qubit has a simple and elegant geometric structure. In this section, we will look at the geometry of the qubit state space and then move on to a system of two qubits.

1.3.1 Geometry of Qubit State Space

The state space of a single qubit is the set of all density operators on Hilbert space $\mathbb{C}^2$. Therefore it is a convex subset of the space of Hermitian operators $\mathsf{Herm}(\mathbb{C}^2)$, which is a four-dimensional vector space. One basis set for $\mathsf{Herm}(\mathbb{C}^2)$ is given by the two dimensional identity matrix and the three Pauli operators as follows:

$$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Any qubit state ρ can be written as a linear combination of this basis set as the following : $\rho = (c_0\mathbb{I} + c_1\sigma_x + c_2\sigma_y + c_3\sigma_z)$, where $c_i \in \mathbb{C}\ \forall\ i$. By putting the conditions to be a valid density operator, we obtain that an arbitrary qubit state can be written as

$$\rho = \frac{1}{2}(\mathbb{I} + n_x\sigma_x + n_y\sigma_y + n_z\sigma_z) = \frac{1}{2}(\mathbb{I} + \vec{n}.\vec{\sigma}), \tag{1.2}$$

where $\vec{n} \equiv (n_x, n_y, n_z)$ is a real vector in $\mathbb{R}^3$ with $|\vec{n}| \leq 1$. This vector is called the Bloch vector for the state ρ. Here $\vec{n}.\vec{\sigma} \equiv n_x\sigma_x + n_y\sigma_y + n_z\sigma_z$ and $\vec{\sigma} \equiv (\sigma_x, \sigma_y, \sigma_z)$. This representation of an arbitrary qubit state has an elegant geometrical interpretation. Each ρ is in a one to one correspondence with its Bloch vector (n_x, n_y, n_z), which satisfies $|\vec{n}| \leq 1$, where the equality holds only for pure states. All the points inside the sphere correspond to mixed qubit states. A pure state

$|\psi\rangle$, located on the surface of the sphere at the (θ,ϕ) point (where $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$) on the surface of the Bloch sphere (see fig.1.1), can be expressed as

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle . \tag{1.3}$$

Here θ is the polar angle from Z axis, and ϕ is the azimuthal angle from X axis.

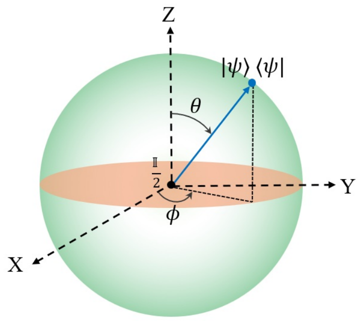


FIG. 1.1 The positions of a pure and a mixed state on the Bloch sphere are shown. Points corresponding to pure states are situated on the surface of the Bloch sphere. The angles θ and ϕ stand for the usual polar and azimuthal angles respectively. The center points correspond to the Maximally mixed state $\mathbb{I}/2$.

1.3.2 Two-Qubit System and Entanglement

Let us consider the simplest example of the composite system corresponding to two spin $1/2$ particle *i.e.* two-qubit systems. The Hilbert space associated with this composite system is $\mathbb{C}^2 \otimes \mathbb{C}^2$. Now let us consider the following state in $\mathbb{C}^2 \otimes \mathbb{C}^2$

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B)$$

where $\{|0\rangle_A \otimes |0\rangle_B, |0\rangle_A \otimes |1\rangle_B, |1\rangle_A \otimes |0\rangle_B, |1\rangle_A \otimes |1\rangle_B\}$ are the orthonormal basis of the composite Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$. Interestingly, this state cannot be decomposed in the form of eq.1.1, which implies $|\psi\rangle_{AB}$ is an entangled state. Any arbitrary two-qubit state $\rho_{AB} \in \mathscr{D}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ can be written in the Hilbert-Schmidt representation as following

$$\rho_{AB} = \frac{1}{4}\left(\mathbb{I}_A \otimes \mathbb{I}_B + \hat{r}.\vec{\sigma}_A \otimes \mathbb{I}_B + \mathbb{I}_A \otimes \hat{s}.\vec{\sigma}_B + \sum_{i,j=1}^{3} t_{ij}\sigma_{i_A} \otimes \sigma_{j_B}\right) \tag{1.4}$$

Here $\mathbb{I}$ is the identity operator, σ_i are the Pauli operators acting on $\mathbb{C}^2$ and $\hat{r},\hat{s} \in \mathbb{R}^3$ with $\hat{m}.\vec{\sigma} = \sum_i m_i\sigma_i$. $\hat{r}$ and $\hat{s}$ are the block vector of respective subsystems and $t_{ij} = \mathrm{Tr}((\sigma_{i_A} \otimes \sigma_{j_B})\rho_{AB})$ are the elements of the correlation matrix T. Also, for ρ_{AB}

to be a valid density operator, the following conditions must hold:

$$|\hat{r}|^2 + |\hat{s}|^2 + \sum_{i,j=1}^{3} t_{ij}^2 \leq 3. \tag{1.5}$$

Subsystem of a Composite System: If we know the state of a composite system, we can determine the marginal state of any of its subsystems. Suppose ρ_{AB} is the density operator of a bipartite system of two subsystems, say A and B. The reduced density operator for subsystem A is defined by

$$\rho_A \equiv \mathbf{Tr}_B[\rho_{AB}]$$

where Tr_B is known as partial trace over subsystem B, which is a linear map from $\mathscr{D}(\mathscr{H}_A \otimes \mathscr{H}_B)$ to $\mathscr{D}(\mathscr{H}_A)$. Using the above definition, it is easy to find that the marginal quantum states of both the subsystems in the composite state $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B)$ is $\frac{\mathbb{I}}{2}$, which is the maximally mixed state of a qubit.

1.4 Information Theory

Information theory is the study of quantification, storage, and transmission of information in terms of different physical systems. Entropy is a crucial element of information theory, which measures the amount of uncertainty in any physical system. Information can be stored and transferred using any physical system, e.g. classical system, quantum system, etc.

1.4.1 Classical Information and Entropy

Entropy captures the notion of how much information we gain in learning the value of a random variable $\mathscr{X}$. The entropy of a random variable is defined as a function of the probabilities of different values and is independent of the labels used for those values. Let the outcome $x_i \in \mathscr{X}$ occur with probability P_i, then the Shannon entropy of the random variable is given by

$$\mathrm{H}(\mathscr{X}) \equiv -\sum_{i=1}^{N} P_i \log_2 P_i \tag{1.6}$$

where N is the cardinality of $\mathscr{X}$. It can be seen from the definition that Shannon entropy ($\mathrm{H}(\mathscr{X})$) is a concave function *i.e.* $\mathrm{H}(\lambda\mathscr{X} + (1-\lambda)\mathscr{Y}) \geq \lambda\mathrm{H}(\mathscr{X}) + (1-\lambda)\mathrm{H}(\mathscr{Y})$. In the case of a binary system *i.e.* $N = 2$ the entropy reaches the maximal value $\mathrm{H}(\mathscr{X}) = 1$ for the output probability distribution $(\frac{1}{2}, \frac{1}{2})$. It implies that 1 bit of information is needed to convey the outcome of a completely random binary system, e.g. unbiased two-face coin. Moreover, this Shannon entropy has a great implication for the Source Coding Theorem.

Source Coding Theorem: Let an N letter asymptotic message be encrypted by the letters $\{x_i\}_{i=1}^N$ following independent identical probability distribution (i.i.d) $\{P_i\}_{i=1}^N$. There always exists a most suitable scheme for coding that takes on an average $\mathrm{H}(\mathscr{X})$ bit/letter [1].

The joint probability distribution of two random variables $\mathscr{X}$ and $\mathscr{Y}$ is given by $P(x,y)$ where, $x \in \mathscr{X}$ and $y \in \mathscr{Y}$. Their joint entropy is given by

$$\mathrm{H}(\mathscr{X},\mathscr{Y}) \equiv \sum_{i=1}^{|\mathscr{X}|} \sum_{j=1}^{|\mathscr{Y}|} -P(x_i,y_j)\log_2 P(x_i,y_j). \tag{1.7}$$

If there exists a correlation between the random variables $\mathscr{X}$ and $\mathscr{Y}$, then the global uncertainty is strictly less than the summation of individual uncertainty *i.e.* $\mathrm{H}(\mathscr{X},\mathscr{Y}) < \mathrm{H}(\mathscr{X}) + \mathrm{H}(\mathscr{Y})$. The correlation of two random variables is captured by the mutual information, which is defined as

$$\mathrm{I}(\mathscr{X} : \mathscr{Y}) = \mathrm{H}(\mathscr{X}) + \mathrm{H}(\mathscr{Y}) - \mathrm{H}(\mathscr{X},\mathscr{Y}).$$

For instance, if we consider a six face dice $\mathscr{D} = \{1,2,3,4,5,6\}$ and a two outcome coin $\mathscr{C} = \{H,T\}$ with a joint distribution such that the joint outcomes $\{1H,2H,3H,4T,5T,6T\}$ occur with probability $\frac{1}{6}$. It is clear from the above definition of mutual information that $\mathrm{I}(\mathscr{D} : \mathscr{C}) = 1$, which implies that knowing the outcome of one random variable will reveal 1-bit information about the other variable. Moreover, this mutual information has huge implications for the channel coding theorem [1].

Channel Coding Theorem: In an N letter message where N is asymptotically large, encrypted with the letters in $\mathscr{X} = \{x_i\}_{i=1}^{|\mathscr{X}|}$ ($\mathscr{X}$ is a random variable) following independent and identical probability distribution $\{P_i\}_{i=1}^{|\mathscr{X}|}$ passing through a noisy channel N and becomes $\{y_i\}_{i=1}^{|\mathscr{Y}|} \in \mathscr{Y}$ (Y is a random variable) following independent and identical probability distribution $\{P_i\}_{i=1}^{|\mathscr{Y}|}$. Then, there exists a reliable coding scheme that must be optimal to compress the message in I(X : Y) bits/letter [1].

1.4.2 Quantum Information and Entropy

The concept of entropy for quantum systems was given by Von Neumann in 1932, much before Shannon, in an attempt to build a connection with statistical or thermodynamical entropy. As mentioned earlier, quantum systems are described by a density operator, and von Neumann entropy is a well-defined function on it. Suppose a state ρ is prepared, then the entropy associated with that state is given by

$$S(\rho) = -\mathrm{Tr}\{\rho \log \rho\}.$$

This quantum entropy is connected with the eigenvalue of the density operator ρ for instance, consider the density operator ρ with the following spectral decomposition $\rho = \sum_i P_i |i\rangle\langle i|$. The quantum entropy $S(\rho) = -\mathrm{Tr}\{\rho \log \rho\}$ of the system ρ is same as the classical Shannon entropy $H(P)$. In case of pure state, the entropy vanishes and it reaches the maximum for maximally mixed states. The entropy remains invariant under unitary operation *i.e.* $S(\rho) = S(U\rho U^\dagger)$ and also shows concavity property *i.e.* $S(\sum_i P_i\rho_i) \geq \sum_i P_i S(\rho_i)$.

The von Neumann entropy plays a significant role in quantum information theory, similar to the role of classical entropy in the classical source coding theorem. In 1995, Benjamin Schumacher provided a key interpretation of quantum entropy, viewing it as the rate of data compression when information is encoded in a quantum system's ensemble. This concept is encapsulated in the *quantum noiseless coding theorem*, which is stated as follows:

Quantum Noiseless Coding Theorem: Given a quantum ensemble $\{|\Psi_i\rangle\}_{i=1}^{|\mathscr{X}|}$ for an n-letter quantum message (with $n \to \infty$) and corresponding probabilities $\{p_i\}_{i=1}^{|\mathscr{X}|}$, there exists an optimal coding scheme that compresses the information to $S(\rho)$ bits per letter while maintaining reliable transmission. Here, $\rho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$ is the density matrix of the ensemble [5].

Quantum noiseless coding theorem gives a lower bound on the amount of quantum system required for reliable transformation of quantum messages from Alice to Bob. But the most fundamental quantity in information theory is the amount of mutual information $H(\mathscr{X} : \mathscr{Y})$ achieved by sending any physical system, where $\mathscr{X}$ and $\mathscr{Y}$ are Alice and Bob's Random variable respectively. Holevo gave the bound on maximal mutual information for a fixed encoding of Alice using quantum systems.

Holevo Bound: Suppose Alice prepare a state $\rho_{\mathscr{X}}$ where $\mathscr{X} = \{1, 2, \cdots, n\}$ with probabilities $p_1, p_2, \cdots, p_n$. Bob performs a measurement described by POVM elements $\{E_y\}_{y=1}^m$ on the state with measurement outcome $\mathscr{Y}$. The Holevo bound states that for any such measurement, Bob may do

$$\mathrm{H}(\mathscr{X}:\mathscr{Y}) \leq \mathrm{S}(\rho) - \sum_x p_x \mathrm{S}(\rho_x) \tag{1.8}$$

where $\rho = \sum_x p_x \rho_x$. It implies that 1 qbit cannot be used to send more than 1-cbit information [23].

1.5 Quantum correlation

Quantum correlations are explored from various angles, starting from nonlocality, device-independent certification, and communication complexity, with nonlocality being a central focus. In the context of nonlocality, quantum correlations are often treated in a device-independent manner. This approach views quantum correlations as the set of input-output statistics obtained from local measurements on a bipartite quantum system, which we will focus on for now. These input-output statistics are denoted by $P(ab|xy) = \mathrm{Tr}\left[(E_x^a \otimes E_y^b)\rho_{AB}\right]$, where E_x^a and E_y^b represent the POVM (Positive Operator-Valued Measure) elements corresponding to the measurement outcomes and ρ_{AB} is the shared quantum state between the two parties.

1.5.1 Bell Nonlocality of Quantum correlation

In the most general Bell theorem, two particles are with two parties separated in distant labs (say Alice and Bob) and they receive some random classical variable x and y respectively. They perform some measurements on their particle based on the classical input x and y yielding the output a and b respectively. Bell showed that under Locality and reality assumption, the joint probability factorize is as follows

$$P(ab|xy) = \sum_\lambda P(\lambda)P(a|x,\lambda)P(b|y,\lambda) \tag{1.9}$$

In the special case of Bell-CHSH inequality, each party has two measurement choices with dichotomic outcome *i.e.* $x, y \in \{0,1\}$ and $a, b \in \{-1,+1\}$. The Bell-CHSH inequality is given by

$$\mathscr{B}_{CHSH} = |\langle A_0B_0\rangle + \langle A_0B_1\rangle + \langle A_1B_0\rangle - \langle A_1B_1\rangle|$$

where $\langle A_i B_j \rangle = \sum_{a,b} abP(ab|x=i, y=j)$ is expectation value of the measurements. It is easy to check that for any local-Realistic theory

$$\mathscr{B}_{CHSH} \leq 2.$$

Quantum mechanics violates this bound *i.e.* there exists quantum state and measurements on Alice and Bob's side, which will violate the above inequality. For instance, consider the state $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B)$ shared between Alice and Bob and take $A_0 = \sigma_z$, $A_1 = \sigma_x$, $B_0 = \frac{\sigma_z+\sigma_x}{\sqrt{2}}$ and $B_1 = \frac{\sigma_z-\sigma_x}{\sqrt{2}}$. Using Born rule the value of the Bell-CHSH expression becomes $2\sqrt{2}$. Later, Tirelson proved that the maximal violation of Bell-CHSH in quantum is $2\sqrt{2}$ [36]. Bell's theorem is a profound discovery with a deeper implication on the philosophy of nature.

1.5.2 Nosignaling principle and quantum correlation

Quantum correlations exhibit nonlocality but cannot be exploited for faster-than-light communication. To understand this from a general perspective, consider a scenario where Alice and Bob are spatially separated, sharing a bipartite quantum state ρ_{AB}. The marginal density matrix on Bob's side is given by $\rho_B = \mathrm{Tr}_A \rho_{AB}$. Now, if Alice performs a local operation (such as a measurement) E_A on her part of the system, the joint state ρ_{AB} will evolve to

$$\rho'_{AB} = (E_A \otimes \mathbb{I}_B)\rho_{AB} = (M_k \otimes \mathbb{I})\rho_{AB}(M_k^\dagger \otimes \mathbb{I})$$

Where, M_k are the Kraus operator of E_A satisfying $\sum_k M_k^\dagger M_k = \mathbb{I}$. After Alice's operation, the state of Bob's subsystem becomes $\rho'_B = \rho_B$. This implies the reduced density matrix on Bob's side remains unchanged, and no local operation by Alice can alter Bob's subsystem. This ensures that quantum mechanics preserves the no-signaling principle, forbidding superluminal communication.

Let us consider the most general scenario where two separated parties, Alice and Bob, receive random classical variables x and y yielding the output a and b respectively. The correlation will be no-signaling if it follows the following conditions

$$\sum_a P(a,b|x,y) = \sum_a P(a,b|x',y) \quad \forall b,x,x',y \qquad \text{[Alice to Bob NS]}$$
$$\sum_b P(a,b|x,y) = \sum_b P(a,b|x,y') \quad \forall a,x,y,y' \qquad \text{[Bob to Alice NS]}$$

In their pioneering work, Popescu and Rohrlich showed that the no-signaling principle alone is insufficient to fully characterize the set of quantum correlations [37]. In fact, they have given an example of bipartite correlations that, while satisfying the no-signaling principle, cannot be realized within quantum mechanics. This type of super-quantum, no-signaling correlation is known as the Popescu-Rohrlich (PR) box. The PR-box is a bipartite no-signaling probability distribution with binary inputs and outputs for both parties

$$P(a,b|x,y) = \frac{1}{2}\delta_{a\oplus b,xy}.$$

Although the PR-box correlation respects the no-signaling principle, it is not quantum realizable because its Bell-CHSH value reaches the algebraic maximum of 4, thereby exceeding Cirel'son's bound. The existence of such super-quantum, nonlocal correlations that adhere to the no-signaling principle highlights that the set of quantum correlations cannot be derived solely from the no-signaling condition.

1.6 A Brief Outline of the Thesis

In this chapter, we explained the basic ideas and core concepts of quantum mechanics, providing the necessary background for the upcoming chapters. We began by discussing the postulates of quantum theory and used a qubit, the simplest quantum system, as an example to explore some non-classical aspects of quantum correlations. Additionally, we delved into the concept of entropy and its significance in classical and quantum information theory. In the next chapters, we will focus more on various resources and their importance in enhancing the utilities of various communication tasks. The layout of the thesis is as follows:

Chapter 2: In this chapter, we explore the resources that can enhance the utility of various communication tasks and divide them into two categories. Additionally, we examine different types of scenarios related to these communication tasks and how the resources apply in each context.

Chapter 3: Here we mainly focus on the most basic communication scenario known as HFW scenario (Sec. 2.3.1) and explore the role of classical shared randomness in enhancing the utility of classical communication channels by considering a class of two-party games. Additionally, we have also explored the effectiveness of quantum communication as well as some toy theories in this communication task. Our study also covers the noise robustness of quantum encoding and decoding protocols. This chapter is based on our published work [38].

Chapter 4: In this chapter, we are exploring another communication scenario in prepare and measure set-up, also known as the W-ANTV scenario. We have considered the quantum random access code with different compositions of local quantum systems. We have shown that the maximal and minimal composition of local quantum systems are advantageous over quantum composition in some communication task. This chapter is based on our published work [39].

Chapter 5: To harness the communication advantages in real life, we should be able to be certain about the existence of the advantageous resources. In this chapter, we will focus on certifying the maximal composition of local quantum system in a semi-device independent manner. Additionally, we are also studying the foundation implication of these communication resources. This chapter is based on our published works [39, 40].

Chapter 6: In this chapter, we conclude the Thesis by summarizing the findings and discussing possible future directions.

Chapter 2

Effective Resources for different Communication Scenario

2.1 Motivation

In a most general task, either Alice, Bob, or both receive some input, and either one or both provide some output. The referee will declare the payoff for all possible input-output statistics before the game starts. Alice and Bob's aim is to optimize the payoff collectively. Alice and Bob can utilize many resources to maximize their Payoff. These resources can be broadly divided into two categories: the common past resources and the direct communication resources. The payoff will be different based on the availability of other resources belonging to various categories. Depending on the inputs and outputs of Alice and Bob the communication task can be divided into three scenarios *i.e.* Holevo Frenkel Weiner (HFW) scenario, RAC scenario, and Bell scenario. In the next sections, we will discuss the various properties of these resources and their importance in different communication scenarios.

2.2 Types of resources

The resources available to Alice and Bob can be classified into two broad categories.

2.2.1 Type-1 (Common-past resources)

Alice and Bob meet in their common past before the game starts and share some correlated systems, as these resources are generated in the common past of Alice and Bob they are termed as common past resources. They may use these resources later to optimize their payoff (see Fig.2.1a) though these resources do not have any communication utility *i.e.* they cannot be used to send information.

In literature, generally, three different kinds of common past resources are considered:

(CP-1) **Classical Shared Randomness:** Alice and Bob can meet at their common past and create classical correlation such that both Alice and Bob receive the same classical variable λ with probability $\mu(\lambda)$. This can be thought of as performing computational basis measurement on the state

$$|\Psi\rangle_{AB} = \sum_{\lambda=0}^{k-1} \mu(\lambda)\, |\lambda\rangle_A \langle\lambda| \otimes |\lambda\rangle_B \langle\lambda|$$

The mutual information between Alice and Bob can quantify the amount of shared randomness.

(CP-2) **Quantum entanglement:** As defined earlier in Eq.1.1 any state $\rho_{AB} \in \mathscr{D}\left(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}\right)$ which cannot be decomposed as convex combination of product states are entangled. So entangled states are defined as the negation of separable states *i.e.*

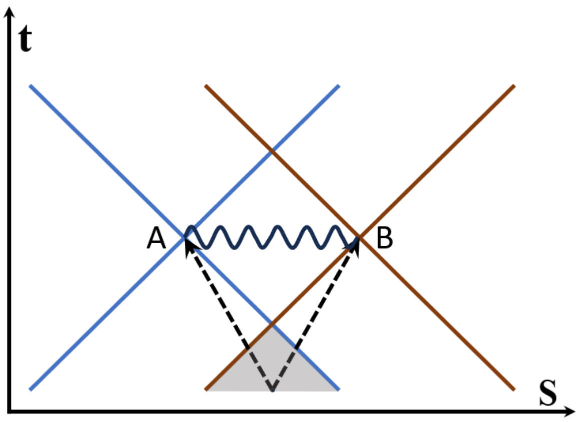
$$\rho_{AB}^{ent} \neq \sum_{\lambda=0}^{k-1} \mu(\lambda) \rho_A^{\lambda} \otimes \rho_B^{\lambda}.$$

Entangled states can yield puzzling 'nonlocal' correlations which cannot be obtained from classical shared randomness [41]. In the case of bipartite pure state von Neumann entropy of the reduced state is a measure of entanglement.

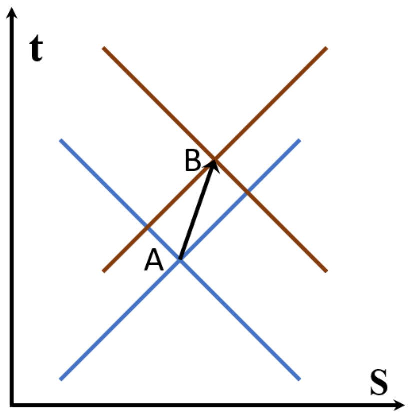
(CP-3) **Beyond quantum nonlocal correlations:** Alice and Bob can create any nosignaling correlation in their common past *e.g.* Popescu-Rohrlich(PR) correlation [37]. The PR correlation is given by

$$\begin{aligned} P(a,b|x,y) &= \frac{1}{2} \quad , a \oplus b = xy \\ &= 0 \quad , \text{otherwise} \end{aligned}$$

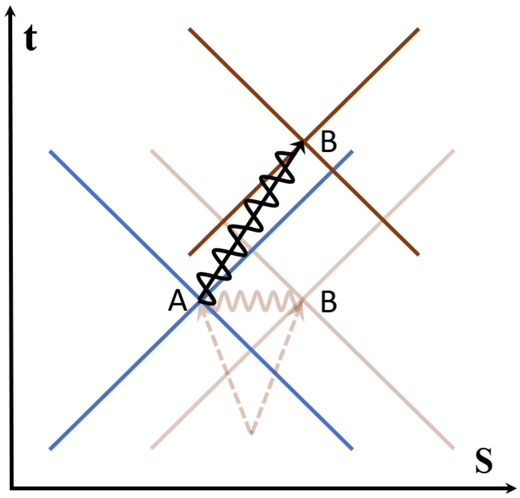
where $a,b,x,y \in \{0,1\}$. The nonlocality of any nosignaling correlation can be measured by the amount of violation of some suitable bell inequality.



(A) Common past resources (Type-1)



(B) Direct communication resource (Type-2)



(C) Combination of Direct and Common past resources

FIG. 2.1 Causal structure of different communication resources. (a) The Shaded region is in the common past of both Alice and Bob. They can create some correlated system like shared randomness, entanglement, etc, which can be used later however their communication capacity is zero. These are known as Common past resources (**Type-**1). (b) Alice is in the causal past of Bob, Alice sends some physical systems (e.g., classical bits, quantum bits, etc.), and Bob receives them at a later time. These are known as Direct communication resource(**Type-**2) and all of them has some nonzero communication capacity. (c) This represents the combination of the two scenarios mentioned earlier. In the beginning, they establish a shared resource (**Type-**1) in their common past. Then Alice sends a physical system (**Type-**2) that is received by Bob in his subsequent causal future.

2.2.2 Type-2 (Direct-communication resources)

Alice takes some physical system and encodes her message in different states of the system (see Fig.2.1b). She then sends the system to Bob, who may perform some measurements to obtain an outcome and optimize the required payoff accordingly. Here we will only consider the scenarios where the physical systems can be sent perfectly without any disturbance from Alice to Bob. Depending on the type of the physical system the direct communication resources can be divided into the following three categories

(DC-1) **Classical Communication:** Alice encodes her inputs into different classical systems and send them to Bob. This encoded classical system could be N distingushable object *i.e.* N different coloured balls or N different symbols. The state space of N dimensonal classical systems is described by simplex with N extreme points. $\log_2 N$ bits of information can be sent by encoding in N different classical systems.

(DC-2) **Quantum Communication:** In case of quantum communication Alice encodes the inputs in quantum systems (Sec. 1.3). The state space of quantum systems is given by $\mathscr{D}(\mathbb{C}^d)$. Alice can encode the inputs in a superposition state in quantum communication. Despite the availability of infinite choice of encoding and decoding the total amount of information is bounded to $\log_2 N$ in N dimensional quantum communication.

(DC-3) **Generalized Communication:** In generalised communication Alice can encodes the inputs in hypothetical systems of generalised probability theory (GPT). This hypothetical system has a convex state space in $\mathbb{R}^N$ [28], *e.g.* polygon systems [42]. Operational dimension of a GPT is defined as the maximal cardinality of set of states which are perfectly distingushable by a single measurement. Encoding the inputs in a GPT with measurement dimension N Alice can send $\log_2 N$ bits of information to Bob.

However, one can consider a communication scenario where both **Type-**1 and **Type-**2 resources are used together. The diagram (see Fig.2.1c) above depicts the causal structure of both Direct communication resources and Common past resources when used in combination.

2.3 All the variant of communication task

In any communication task, different parties receive inputs from the referee, and, as a result, some or all of them have to provide an outcome. The communication scenario varies depending on which party is receiving input and providing output. In the following, we will explain the different communication scenarios in detail.

2.3.1 Holevo Frenkel Weiner Scenario

This is the most basic scenario involving two-party communication. Although the study of communication using classical systems within this scenario originally started in the seminal work of Shannon [1], in the quantum domain it was introduced by Holevo (H) to understand the limitations of information transmission by a quantum channel [23]. More recently, Frenkel & Weiner (FW) have proposed a generalization of this framework [24]. Suppose, Alice and Bob are two distant parties. A Referee provides Alice some classical random variable $x \in \mathscr{X}$, while Bob needs to generate some classical random variable $b \in \mathscr{B}$ and return it to the Referee, according to the correlation $P(b|x)$ generated by Alice and Bob refree gives them some payoff $\beta : P(\mathscr{B}|\mathscr{X}) \to \mathbb{R}$ (see Fig.2.2). The payoff can be defined on a set of correlations satisfying some constrain or on the achievability of some particular correlation.

While none of the **Type-**1 resources has communication utility on their own in the H-FW scenario, to create a resource of the kind CP-1 Alice and Bob must communicate with each other if they do not meet in their common past. On the other hand, the effects of the resource of kind CP-2 can be simulated with sufficient classical communication between Alice and Bob [43]. However, the resource of the kind CP-2, *i.e.* quantum entanglement, cannot be created with local operations and classical communication (LOCC) between Alice and Bob [41]. They can only possess it by preparing it in their common past or if some third agent gives it.

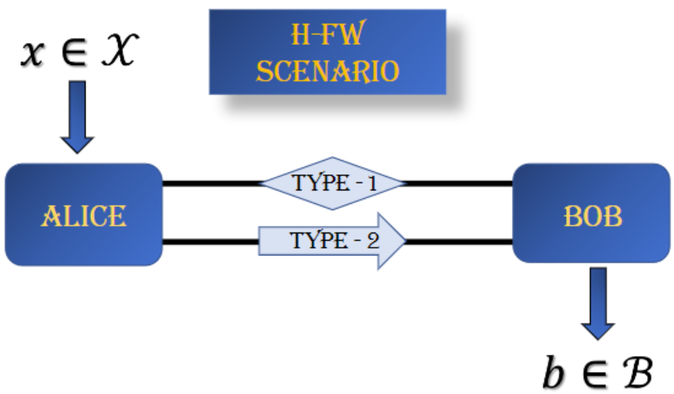


FIG. 2.2 Holevo-Frenkel-Weiner (H-FW) communication scenario. Only Alice is given some classical input $x \in \mathscr{X}$, while only Bob has to generate a classical output $b \in \mathscr{B}$. Two different types of resources - **Type-**1 and **Type-**2 - might be utilized to optimize the targeted payoff $\beta : P(\mathscr{B}|\mathscr{X}) \to \mathbb{R}$.

Although **Type-**1 resources have zero communication utility, they can enhance the communication utility of the **Type-**2 resources. The seminal example is the 'quantum superdense coding' protocol, which shows that preshared entanglement between Alice and Bob can double the classical communication capacity of a perfect quantum channel [7] (see [44–46] for experimental implementations of the protocol). Later, it was shown that entanglement can increase the classical capacity of some noisy quantum channels by an arbitrarily large constant factor over their best-known classical capacity achievable without entanglement [47]. Very recently, it has been shown that entanglement can also enhance the communication utility of a perfect classical channel [48, 49], within this H-FW scenario. Furthermore, the assistance of post-quantum correlation in this case turns out to be more useful than quantum entanglement. Here we note that the success probability of the game proposed in [49] with the resource of 1 bit of classical communication along with some generic no signaling (NS) correlation scales linearly with the famous Clauser-Horne-Shimony-Holt (CHSH) expression [50].

In this H-FW communication scenario, the celebrated no-go result by Holevo limits the information capacity of an n-level quantum system by the optimal value achievable with an n-level classical system [23]. While in Holevo's theorem the utility is calculated through an entropic quantity, namely the mutual information $I(\mathscr{X}:\mathscr{B})$ between Alice's input random variable and Bob's output random variable, Frenkel & Weiner evaluate how successfully Alice and Bob can manage to store and recover the value of $x \in \mathscr{X}$ by requiring Bob to specify a value $b \in \mathscr{B}$ and giving a generic reward of value $f(x,b)$ to them. They have shown that whatever the probability distribution of x and the reward function f are, when using a quantum n-level system, the maximum expected reward obtainable with the best possible team strategy is equal to that obtainable with the use of a classical n-level system [24]. Furthermore, like *Shannon's Noisy Channel Coding Theorem*, Holevo's theorem captures the channel's capacity of reliable transmission rate in the asymptotic limit, while with a single use of the channel, things can go differently. Therefore, the result of Frenkel & Weiner should be seen as an independent no-go theorem while evaluating the communication utility of a quantum system.

It is important to note that Frenkel & Weiner, while deriving their no-go result, consider classical shared randomness (CP-1 resource of **Type-**1) as free, *i.e.* Alice and Bob can have arbitrary amount of shared randomness in their possession. We have already seen that CP-2 and CP-3 resources of **Type-**1 can empower communication utility of different resources of **Type-**2 [7, 44–49]. It is, therefore, a natural question to analyze the communication utility of different resources of **Type-**2 in absence of CP-1 resource of **Type-**1, *i.e.* shared randomness. Precisely this question is the issue we will address in Chapter 3.

2.3.2 W-ANTV (RAC) Scenario

This scenario is more involved and also distinct from the earlier scenario. In this case, the Referee provides random variables $x \in \mathscr{X}$ and $y \in \mathscr{Y}$ to Alice and Bob, respectively, and only Bob needs to return an output random variable $b \in \mathscr{B}$. Accordingly, they will be given some payoff depending on the correlation generated by them $\beta : P(\mathscr{B}|\mathscr{X},\mathscr{Y}) \to \mathbb{R}$. The famous no-go theorem of Holevo [23] later generalized by Frenkel and Weiner [24] is no longer applicable in this scenario. An N-level quantum system can be advantageous over an N-level classical system in this W-ANTV scenario. Random Access Codes (RACs) are canonical examples of tasks that fit perfectly within this scenario. In RAC task $\mathscr{Y}$ is considered to be a set of queries regarding $\mathscr{X}$ (see Fig.2.3). In a RAC task, the sender encodes a data set – typically a string of input bits – onto a physical system of bounded dimension and transmits it to the receiver, who then attempts to guess a randomly chosen part of the sender's data set – typically one of the sender's input bits. While the protocol was first introduced by Wiesner (W) by the name conjugate coding [25], the study drew renewed interest nearly two decades later when it was rediscovered by Ambainis, Nayak, Ta-Shma, and Vazirani (ANTV) [26, 51]. Interestingly, in this task, the communication utility of a qubit (DC-2 resource of **Type-**2) has been shown to be more than a classical bit (DC-1 resource of **Type-**2), even in absence of any kind of **Type-**1 resources [25, 26, 51]. Subsequently, several variants of this task have been studied leading to interesting implications[52–56]. Furthermore, in this scenario, it has also been shown that a resource of **Type-**1 can empower the communication utility of a resource of **Type-**2 [57–59].

Parity Oblivious Random Access Cose (PORAC) is another prominent example of this scenario. Similar to the RAC task here also Bob needs to guess some information about Alices input bit strings, but the amount of communication is no longer restricted. There is a restriction on the type of communication *i.e.* communication must not contain the parity information of the input bits. Spekkens *et. al.* has given bound on the PORAC success for any noncontextual theory and also shown that quantum mechanics violate this bound [60]. There are other variants of this task with deeper foundational implications. In W-ANTV

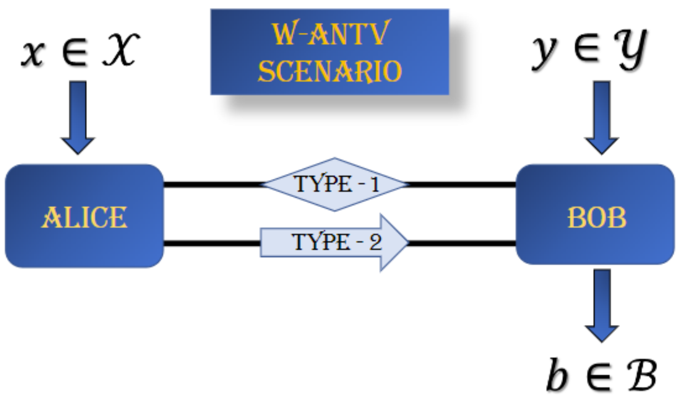


FIG. 2.3 Wiesner-Ambainis- Nayak-Ta-Shma-Vazirani (W-ANTV) communication scenario. While Alice is given some classical input $x \in \mathscr{X}$, Bob is also given (generally some query regarding $\mathscr{X}$) input $y \in \mathscr{Y}$. However, only Bob has to generate a classical output $b \in \mathscr{B}$. Like the H-FW case, two different types of resources might be used to optimize the payoff $\beta : P(\mathscr{B}|\mathscr{X},\mathscr{Y}) \to \mathbb{R}$.

scenario, it has also been shown that post-quantum nonlocal correlations can empower classical communication arbitrarily, making communication complexity trivial [61, 19, 29].

2.3.3 BELL Scenario

In this case both Alice and Bob are given classical random variables $x \in \mathscr{X}$ and $y \in \mathscr{Y}$, respectively; and both of them have to return some random variable $a \in \mathscr{A}$ and $b \in \mathscr{B}$, respectively to the Referee. Accordingly, they are given some payoff $\beta : P(\mathscr{A},\mathscr{B}|\mathscr{X},\mathscr{Y}) \to \mathbb{R}$ (See Fig. 2.4). Unlike the earlier two scenarios, they are not allowed to communicate with each other, *i.e.* resources of **Type-**2 are prohibited in this case. However, they can share some resources of **Type-**1. This scenario is mostly studied within the quantum foundations community to understand the strength of different correlations. The seminal result of Bell [14, 15] (see also [62]) establishes that this scenario is capable of revealing a hierarchy among different common past resources. In fact, from entangled quantum states, one can come up with correlations that yield more success for some suitably chosen payoffs than any classical shared randomness. Such correlations are more popularly known as Bell-nonlocal correlations (See Sec.1.5.1). Subsequently, it has been shown that this scenario is capable of making a distinction between quantum and post-quantum correlations [37, 36].

More recently, this framework has been proven to be useful for separating different mathematical models for composite systems [63] which accordingly

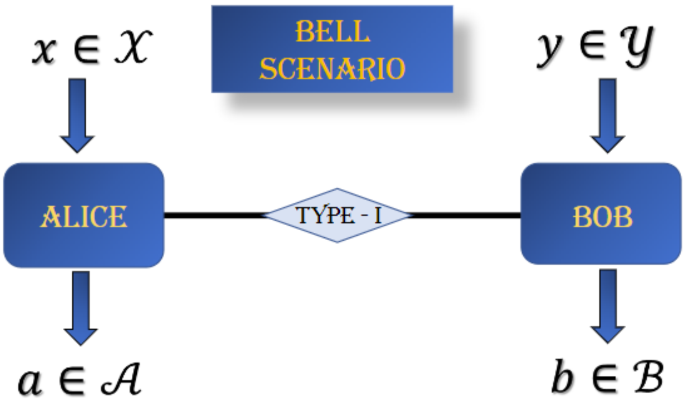


FIG. 2.4 Canonical Bell Scenario. Here, both Alice and Bob are given inputs $x \in \mathscr{X}$ and $y \in \mathscr{Y}$, respectively. Both of them need to return back classical outputs $a \in \mathscr{A}$ and $b \in \mathscr{B}$. Unlike the earlier two cases, no communication is allowed between Alice and Bob. However they can utilize **Type-**2 resources to optimize their joint payoff $\beta : P(\mathscr{A}, \mathscr{B}|\mathscr{X}, \mathscr{Y}) \to \mathbb{R}$.

provides answers to some long-standing problems in complexity theory and operator theory [64] and leads to some undecidable consequences regarding the structure of quantum logic [65]. More recently, this scenario has been further generalized where Alice and Bob are given quantum inputs instead of classical random variables [66, 67]. In Chapter 5 we will certify different useful communication using a quantum input bell scenario.

Chapter 3

Communication advantage in Holevo-Frankel-weinar Scenario

3.1 Introduction

The simplest communication scenario, known as HFW scenario involves two distant parties – a sender (Alice) and a receiver (Bob) – where Alice aims to transmit some message to Bob by sending some physical systems (see Sec.2.3.1). The pioneering 'quantum superdense coding' protocol establishes a quantum advantage by showing that quantum entanglement, preshared between sender and receiver, can double the classical communication capacity of a perfect qubit channel [7]. This is quite striking, as in such a scenario, quantum entanglement on its own has no communication utility, and according to the fundamental no-go theorem of Holevo [23], the communication capacity of a perfect qubit channel alone is no more than the capacity of a perfect one-bit classical channel. Recently, the no-go implication of Holevo has been strengthened further by Frenkel & Weiner while evaluating the communication utility of a quantum system in absence of preshared entanglement [24]. While in Holevo's theorem, communication utility is measured in terms of the mutual information between the random variable the sender intends to send and the random variable the receiver obtains after the channel action, Frenkel & Weiner quantify a channel's utility through a generic reward function rather than only mutual information and still establish that the communication utility of an n-level quantum system is the same as that of an n-level classical system.

The contribution of this chapter starts with the following pivotal observation: while quantum entanglement and classical shared randomness both have zero communication utility on their own, the no-go results in [23, 24] consider the former to be a costly resource, and hence not allowed to be preshared between the parties, whereas the latter is freely available between them. This assumption is supported by the general practice within the study of Bell's nonlocality [14, 15, 68], wherein classical shared randomness is considered to be free as they result only in 'local' correlations, while quantum entanglement can result in puzzling 'nonlocal' correlations [69–71]. However, it is important to note that to create classical shared randomness between two distant parties, classical communication is a necessary resource, and hence in this work we will consider it to be a costly resource. A number of works in other branches of research [72–74] as well as in quantum information theory [75–78] already exist where nontrivial utilities of classical shared randomness have been pointed out.

3.2 Classical superdense coding

In this section, we will show how classical shared randomness can play a nontrivial role in enhancing the communication utility of a perfect classical communication channel. To this aim, we propose a class of two-party communication games and then analyze the payoffs of these games, first with 1-bit of classical communication only and then with the additional assistance of shared randomness.

3.2.1 A two-party communication game

Suppose Charlie is the manager of a chain of restaurants. Let the total number of restaurants be n. On each day one of the restaurants remains closed and which one is to be closed is decided by Charlie randomly with uniform probability. Charlie informs Alice which restaurant is closed for the day. Another fellow, Bob, must visit one of Charlie's restaurants for lunch each day. However, Bob does not know which restaurant is closed for the day. He relies on Alice to communicate that information to him. Alice is restricted to communicating only 1-bit of information to Bob. The collective aim of Alice and Bob is to ensure that the following two conditions are satisfied:

(h1) Bob never visits a closed restaurant.

(h2) Bob visits each restaurant with equal probability.

The situation can be described with the following 'visit' matrix:

$$\mathbb{V} \equiv \begin{array}{c} \\ 1_c \\ 2_c \\ . \\ . \\ n_c \end{array} \begin{array}{c} \begin{array}{cccc} 1_b & 2_b & \cdots & n_b \end{array} \\ \begin{pmatrix} p(1_b|1_c) & p(2_b|1_c) & \cdots & p(n_b|1_c) \\ p(1_b|2_c) & p(2_b|2_c) & \cdots & p(n_b|2_c) \\ . & . & \cdots & . \\ . & . & \cdots & . \\ p(1_b|n_c) & p(2_b|n_c) & \cdots & p(n_b|n_c) \end{pmatrix} \end{array} \tag{3.1}$$

The entry $p(i_b|j_c)$ represents Bob's probability of visiting the i^{th} restaurant (hence the subscript 'b') given that the j^{th} restaurant is closed (hence the subscript 'c'). The condition (h1) implies

$$p(i_b|i_c) = 0, \ \forall\, i \in \{1, \cdots, n\}, \tag{3.2}$$

i.e. all the diagonal entries of the matrix $\mathbb{V}$ must be zero. Bob's probability of visiting the i^{th} restaurant can be obtained from the sum of the entries for the i^{th} column of the matrix $\mathbb{V}$. The condition (h2) thus reads as

$$p(i_b) = \sum_{j=1}^{n} p(i_b|j_c)p(j_c) = \frac{1}{n}\sum_{j=1}^{n} p(i_b|j_c) = \frac{1}{n}. \tag{3.3}$$

Here, $p(j_c) = 1/n, \ \forall\, j \in \{1, \cdots, n\}$, as the closing probabilities are assumed to be uniform and the last equality is imposed by the condition (h2). Alice can send only 1-bit of classical communication to Bob. Furthermore, we assume that shared randomness is a costly resource. Alice and Bob, however, are free to use local randomness on their part while playing the game. With this, we can formally define different classical strategies.

Definition 1 (Classical deterministic strategy)**.** *A classical deterministic strategy is an encoding-decoding tuple* (E, D)*, where* E *is a '*$\log n$*-bit to* 1*-bit' deterministic function and* D *is a '*1*-bit to* $\log n$*-bit' deterministic function, i.e.* $\mathrm{E} : \{1, \cdots, n\} \mapsto \{0, 1\}$ *and* $\mathrm{D} : \{0, 1\} \mapsto \{1, \cdots, n\}$.

Definition 2 (Classical mixed strategy)**.** *A classical mixed strategy is a probabilistic strategy* $(P_{\mathbb{E}}, P_{\mathbb{D}})$*, where* $P_{\mathbb{E}}$ *and* $P_{\mathbb{D}}$ *are probability distributions over the space of deterministic encodings (*$\mathbb{E}$*) and decodings (*$\mathbb{D}$*), respectively.*

Definition 3 (Classical correlated strategy)**.** *A classical correlated strategy is a probability distribution $P_{\mathbb{E}\times\mathbb{D}}$ over the space of Cartesian product of deterministic encodings and decodings.*

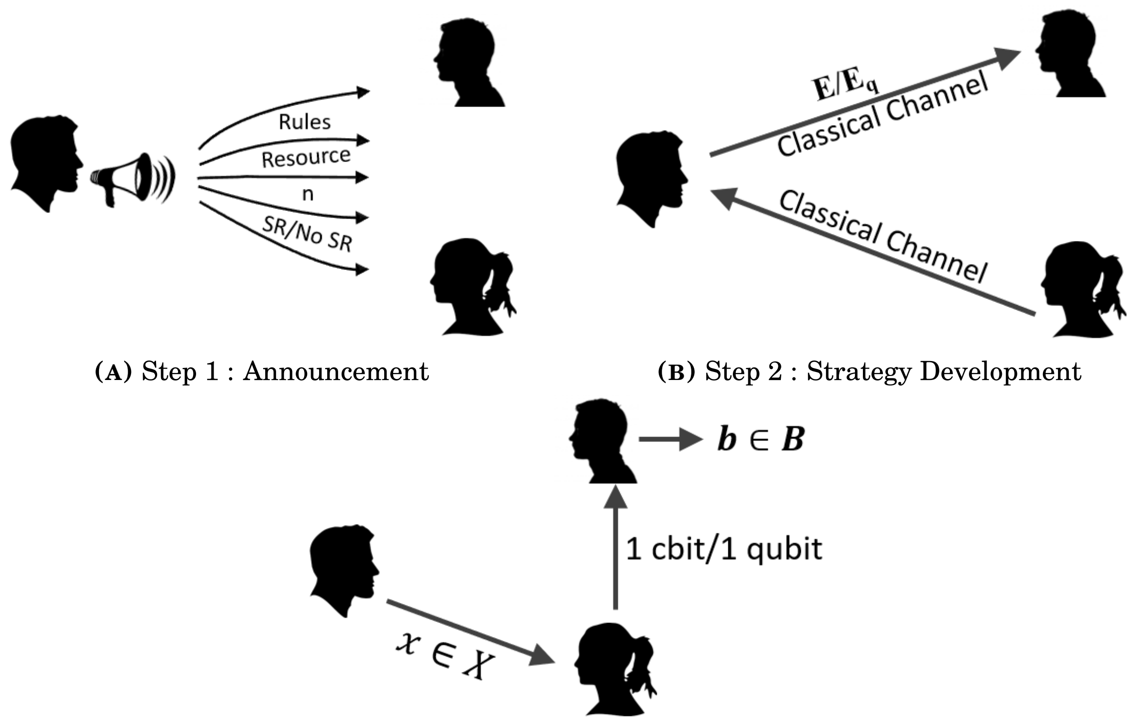
Classical mixed strategies can be realized with local randomness on Alice's and Bob's sides. On the other hand, the implementation of classical correlated strategies requires Alice and Bob to possess classical shared randomness. Formally, classical shared randomness between two parties can be defined as a joint probability distribution $\{p(\alpha,\beta)\}$ on $\mathtt{A}\times\mathtt{B}$, where $\alpha\in\mathtt{A}$ and $\beta\in\mathtt{B}$ are random variables possessed by Alice and Bob, respectively. The amount of shared randomness can be quantified through classical mutual information, $I(\mathtt{A}:\mathtt{B}):=H(\mathtt{A})+H(\mathtt{B})-H(\mathtt{A},\mathtt{B})$, where $H(\mathtt{W}):=-\sum_w p(w)\log p(w)$ is the Shannon entropy of the random variable $w\in\mathtt{W}$. It is not hard to see that if Alice and Bob share the classically-correlated state $\rho=\frac{1}{2}\left|00\right\rangle\left\langle 00\right|+\frac{1}{2}\left|11\right\rangle\left\langle 11\right|\in\mathscr{D}(\mathbb{C}^2\otimes\mathbb{C}^2)$ then they can extract 1-bit of classical shared randomness by performing σ_z measurement on their respective subsystems. A similar process will yield $H(p)$-bit of shared randomness from the state $\rho=p\left|00\right\rangle\left\langle 00\right|+(1-p)\left|11\right\rangle\left\langle 11\right|$, with $p\in[0,1]$. For more elaborative discussions on classical shared randomness from a resource theoretic perspective we refer to the work [78].

At this point, we digress a bit to discuss mixed and correlated strategies in the light of Bayesian game theory. John Nash, in his seminal work, introduced the concept of Nash equilibrium and proved that any game with a finite number of actions for each player always has a mixed strategy Nash equilibrium [79]. Later, Harsanyi introduced the notion of Bayesian games where each player has some private information unknown to other players [80–82]. Aumann proved that in the Bayesian game scenario the notion of mixed strategy Nash equilibrium needs to be generalized to a correlated equilibrium where some adviser provides advice to the players in the form of shared randomness to achieve the correlated equilibrium [72]. Note that every pure/ mixed Nash equilibrium is also a correlated equilibrium, but the set of correlated equilibria is strictly larger than the set of mixed strategy Nash equilibria. It has also been shown that correlated equilibria are easier to compute [83]. Recently several interesting results have been reported connecting the study of Bayesian game theory and quantum nonlocality [84–87].

In a communication task, the following two questions are important to be explored. Firstly, whether there exists an optimal strategy to perform the task, and secondly, if it does exist, whether it is achievable with the resources available.

For instance in Bayesian games where each player has some private information unknown to the other player although there exist the notion of Nash Equilibrium but its achievability is in question. Rather, a more general concept in this case is the correlated equilibrium which can be achieved if correlations are provided to the players as assistance. In communication scenarios, one player may have partial knowledge of the other player's information due to limited communication from the latter to the former. In this work, we focus on the existence of a winning strategy with classical communication, quantum communication, and the assistance of classical correlation (Shared-randomness). Considering shared randomness as a valuable resource, there may exist winning strategies that require its assistance. We consider the scenario where Charlie, as a referee, provides different resources to Alice and Bob to implement the winning strategy. Charlie allows arbitrarily large classical side channels from Alice to him and also from him to Bob. Through this, Charlie will help Alice and Bob develop the strategy they will follow once the game starts. Apart from strategy development, the establishment of shared randomness between Alice and Bob will not be allowed through these side channels intervened by Charlie. In other words, we can say that the side channels are constrained. For an explicit example, let's consider a specific two-party communication task: the restaurant Game and their winning conditions (h1,h2) and explore the achievability of winning strategies. So the game setup can be divided into three steps: Announcement, Strategy development, and Active Game-play (see Fig.3.1) For an explicit example, let's consider a specific two-party communication task: the restaurant Game and their winning conditions (h1,h2) and explore the achievability of winning strategies. So the Game setup can be divided into three steps: Announcement, Strategy development, and Active Game-play (see Fig.3.1).

- Step 1 : Announcement (see Fig.3.1a)
  a. Charlie is hosting a game where some payoff will be given depending on fulfilling certain winning conditions (h1 & h2).
  b. Two random players who have never met in the past, Sender (Alice) and Receiver (Bob), participate in the restaurant game.
  c. Charlie declares number of restaurant (n) Bob has to visit in future and demands them to win $H_n(1/n)$ game.



(A) Step 1 : Announcement

(B) Step 2 : Strategy Development

(C) Step 3 : Active Gameplay

FIG. 3.1 The restaurant game can be divided into three different steps. (A) In the first step, Charlie announces the rules of the game and also declares the resources that will provided to the players. (B) An arbitrarily large classical side channel is opened from Alice to Charlie and from Charlie to Bob with the constraint that Charlie will forward only the encoding strategies to Bob. Once the encoding strategies have been forwarded, these side channels will be permanently closed. (C) In the active part of the game, Charlie will give $x \in \mathscr{X}$ to Alice and Alice sends 1-cbit or 1-qubit to Bob and accordingly Bob gives an output $b \in \mathscr{B}$.

d. Charlie also declares the communication resources (classical channel/quantum channel) that will be opened in the future and announces whether some shared randomness will be provided as assistance.

- Step 2 : Strategy Development (see Fig.3.1b)

  a. Charlie opens an arbitrarily large constrained classical channel from Alice to Charlie and from Charlie to Bob. But Charlie will communicate to Bob only the encoding strategy received from Alice.

  b. Alice chose an encoding strategy based on available resources and sent the encoding strategies to Bob via Referee.

c. Charlie now closes both communication channels, from Alice to him and him to Bob.

- Step 3 : Active Gameplay (see Fig.3.1c)

  a. Charlie provides $x \in \mathscr{X}$ to Alice.

  b. A one cbit/qubit communication channel has been opened from Alice to Bob.

  c. Bob outputs $b \in \mathscr{B}$ using some suitable decoding strategy.

In this scenario shared randomness is a costly resource as the participating players are previously unknown to each other also they cannot generate shared randomness with the constrained classical channels. This setup explains the achievability of different winning strategies with the assistance of different communication resources but in the rest of the paper, we will focus on the existence of winning strategy. Coming back to the winning strategy in our restaurant game, Alice and Bob can follow any mixed strategy whenever 1-bit of communication is allowed, but a correlated strategy requires additional resource of shared randomness. It is easy to see that the condition (h1) can be readily satisfied within the limited set of mixed strategies. Alice just tells Bob if the first restaurant is open or closed using 1-bit classical message. If the first restaurant is open, Bob visits the first restaurant. Otherwise, he visits one of the other restaurants. However, satisfying the second condition is more tricky. In the next subsection, we analyze the cases where it can be satisfied.

3.2.2 Games winnable with mixed strategies

We can consider a more general class of games where the winning condition (h2) generalizes as follows:

(h1′) Bob never visits a closed restaurant $\equiv$ (h1).

(h2′) Bob visits the k^{th} restaurant with probability $\gamma_k \geq 0,\ \forall\, k \in \{1,\cdots,n\}$, where each γ_k can in general be different.

Values of γ_k's are known to both Alice and Bob before the game starts. Since all restaurants are closed with equal probability, we therefore have $\gamma_k \leq \left(1-\frac{1}{n}\right),\ \forall\, k$. Clearly, each tuple $(\gamma_1,\gamma_2,\cdots,\gamma_n)$ defines a game which we will denote as $\mathbb{H}^n(\gamma_1,\gamma_2,\cdots,\gamma_n)$, and the special case where γ_i's are uniform will be denoted as $\mathbb{H}^n(1/n)$. Our aim is to find which games are winnable under mixed classical strategies. A generic such strategy can be realized in the following steps.

(S-1) If the k^{th} restaurant is closed, Alice tosses a 2-sided biased coin having the outcomes $\{0,1\}$. The outcome probabilities of the coin are given by $P_k(0) = \alpha_k$ and $P_k(1) = 1-\alpha_k$.

(S-2) Alice communicates the outcome of the coin toss to Bob through the perfect 1-bit classical channel.

(S-3) Bob prepares two n-sided coins with outcome probabilities specified by the probability vectors $\vec{r} = (r_1, r_2, \cdots, r_n)$ and $\vec{q} = (q_1, q_2, \cdots, q_n)$, respectively. If he receives 0 from Alice he tosses the $\vec{r}$ coin and visits the i^{th} restaurant if i^{th} outcome occurs. He follows a similar strategy with the $\vec{q}$ coin if 1 is received from Alice.

With this strategy, the conditional probability $p(m_b|k_c)$ that Bob visits the m^{th} restaurant provided the k^{th} restaurant is closed turns out to be,

$$p(m_b|k_c) = \alpha_k \times r_m + (1-\alpha_k) \times q_m. \tag{3.4}$$

To satisfy the first condition that Bob never visits a closed restaurant we must have,

$$p(i_b|i_c) = \alpha_i \times r_i + (1-\alpha_i) \times q_i = 0 \tag{3.5}$$
$$\forall\, i \in \{1, 2, \cdots n\}.$$

Eq.(3.5) holds for the i^{th} restaurant if and only if at least one of the following conditions is satisfied:

$$\alpha_i = 0 \text{ and } q_i = 0, \tag{3.6a}$$
$$\alpha_i = 1 \text{ and } r_i = 0, \tag{3.6b}$$
$$r_i = 0 \text{ and } q_i = 0. \tag{3.6c}$$

Without loss of generality we can divide the set of n restaurants in two set i.e. $\mathbb{P} \equiv \{1, \cdots, a\}$ and $\mathbb{O} \equiv \{a+1, \cdots, n\}$, such that $\gamma_i > 0\ \forall\, i \in \mathbb{P}$ and $\gamma_i = 0\ \forall\, i \in \mathbb{O}$. For every restaurant either Eq.(3.6a) or Eq.(3.6b) or Eq.(3.6c) holds. If $i \in \mathbb{P}$ then Eq.(3.6c) cannot hold for restaurant i since that would yield $\gamma_i = 0$. Thus for $i \in \mathbb{P}$ either Eq.(3.6a) or Eq.(3.6b) must hold. Also we can conclude that for a restaurant $i \in \mathbb{O}$ Eq.(3.6c) must hold. Therefore, the best they can do with a mixed classical strategy is to make a partition of restaurants into three sets X, Y

and Z such that each of the restaurants in sets X, Y and Z satisfies Eq.(3.6a), Eq.(3.6b) and Eq.(3.6c), respectively. Alice sends 1 if a restaurant from the set X is closed and sends 0 if a restaurant from the set Y is closed. If a restaurant $j \in Z$ is closed, she sends 0 with probability α_j and 1 with probability $1-\alpha_j$. Bob never visits the restaurants in X whenever he receives 1 and never visits the restaurants in Y whenever he receives 0. Formally,

$$
\begin{aligned}
X &\equiv \{j \,|\alpha_j = 0 \text{ and } q_j = 0\}, \\
Y &\equiv \{j \,|\alpha_j = 1 \text{ and } r_j = 0\}, \\
Z &\equiv \{j \,|r_j = 0 \text{ and } q_j = 0\} = \mathbb{O}, \\
X \cup Y &= \{1, 2, \cdots, a\} \;\; \& \;\; X \cap Y = \emptyset.
\end{aligned}
$$

Thus, the probability $p(m_b) = \sum_k p(m_b|k_c)p(k_c)$, that Bob visits m^{th} restaurant turns out to be,

$$
\begin{aligned}
p(m_b) &= \frac{1}{n}\left[\sum_{k\in X} p(m_b|k_c) + \sum_{k\in Y} p(m_b|k_c) + \sum_{k\in Z} p(m_b|k_c)\right] \\
&= \frac{1}{n}\left[\sum_{k\in X} q_m + \sum_{k\in Y} r_m + \sum_{k\in Z} (\alpha_k \times r_m + (1-\alpha_k)\times q_m)\right]. \qquad (3.7)
\end{aligned}
$$

According to the condition (h2′) we have $p(m_b) = \gamma_m$. and depending on whether m belongs to X,Y or Z we have,

$$
\begin{aligned}
\gamma_m &= \frac{1}{n}\left[\sum_{k\in Y} r_m + \sum_{k\in Z} \alpha_k \times r_m\right] \\
&= \left[\frac{|Y|}{n} + \bar{\alpha}_z \frac{|Z|}{n}\right] r_m, \;\; \forall\, m \in X; \qquad (3.8a) \\
\gamma_m &= \frac{1}{n}\left[\sum_{k\in X} q_m + \sum_{k\in Z} (1-\alpha_k)\times q_m\right] \\
&= \left[\frac{|X|}{n} + (1-\bar{\alpha}_z)\frac{|Z|}{n}\right] q_m, \;\; \forall\, m \in Y; \qquad (3.8b) \\
\gamma_m &= 0, \;\; \forall\, m \in Z; \qquad (3.8c)
\end{aligned}
$$

where $|\cdot|$ denotes the cardinality of a set, and $\bar{\alpha}_z := \frac{1}{|Z|}\sum_{k\in Z} \alpha_k$. Thus, a classical winning strategy exists for the given game if and only if there exists a partition of $\mathbb{P}$ into two non-empty sets X and Y, along with two probability vectors $\vec{r}$ and $\vec{q}$

such that

$$r_m = \frac{n}{|Y| + \bar{\alpha}_z |Z|} \gamma_m, \quad \forall \; m \; \in \; X, \tag{3.9a}$$

$$q_m = \frac{n}{|X| + (1 - \bar{\alpha}_z)|Z|} \gamma_m, \quad \forall \; m \; \in \; Y. \tag{3.9b}$$

This defines the class of games that are winnable by classical mixed strategies. From here on-wards we will specifically consider the games with $\gamma_1 > 0 \; \forall \; i$. So it is better to see how Eq.(3.8a)-(3.8c) get modified in such a case. Since we have $|Z| = 0$, no restaurant can lie in the set Z, thus we have

$$\gamma_m = \frac{|Y|}{n} r_m, \quad \forall \; m \in X; \quad \gamma_m = \frac{|X|}{n} q_m, \quad \forall \; m \in Y. \tag{3.10}$$

Note that for the case where $\gamma_i > 0 \; \forall \; i$ sets X and Y must be non empty. Now the probability vectors $\vec{r}$ and $\vec{q}$ are given by

$$r_m = \frac{n}{|Y|} \gamma_m, \; \forall \; m \in X; \quad q_m = \frac{n}{|X|} \gamma_m, \; \forall \; m \in Y. \tag{3.11}$$

Special cases: $\mathbb{H}^n(1/n)$

For this special case we have $|Z| = 0$ and $a = n$. Using $\sum_{m \in X} r_m = 1$ and $\sum_{m \in Y} q_m = 1$ in Eq.(3.11) we have

$$\sum_{m \in X} \gamma_m \frac{n}{|Y|} = 1 = \sum_{m \in Y} \gamma_m \frac{n}{|X|}, \tag{3.12}$$

which for $\mathbb{H}^n(1/n)$ game reduces to

$$\sum_{m \in X} \frac{1}{|Y|} = \sum_{m \in Y} \frac{1}{|X|}, \quad \Rightarrow \quad \frac{|X|}{|Y|} = \frac{|Y|}{|X|}. \tag{3.13}$$

This further implies $|X| = |Y| = n/2$. Thus, $\mathbb{H}^n(1/n)$ cannot be won in the case when n is odd. However if n is even then the task becomes easy. We take $X = \{1, 2, ..., n/2\}$ and $Y = \{n/2+1, n/2+2, ..., n\}$. Alice informs Bob if the closed restaurant belongs to X or Y. If the closed restaurant is among X, then Bob tosses a n/2 -sided fair coin and visits one of the restaurants in Y. If the closed restaurant is in Y, then Bob tosses a n/2-sided fair coin and visits one of the restaurants in X. Thus a $\mathbb{H}^n(1/n)$ game is always winnable using a mixed strategy if and only if n is even.

Three-restaurant general case: $\mathbb{H}^3(\gamma_1, \gamma_2, \gamma_3)$

Note that in the three restaurant case at most one of the γ_i can be 0. All games where one out of the three γ_i's is 0 are winnable with a classical mixed strategy. We provide explicit strategy for such cases. Consider that $\gamma_3 = 0$. Alice sends 0 if restaurant 1 is closed, she sends 1 if restaurant 2 is closed and if restaurant 3 is closed she sends 0 with probability p and 1 with probability $1-p$. Bob visits restaurant 2 if he receives a 0 and he visits restaurant 1 if he receives a 1. This strategy ensures that Bob never visits a closed restaurant and we also have

$$\gamma_1 = \frac{1}{3}[1+(1-p)], \quad \gamma_2 = \frac{1}{3}[1+p], \quad \gamma_3 = 0. \tag{3.14}$$

With an appropriate choice of the values for $p \in [0,1]$ any game with $\gamma_3 = 0$ can be won. Similar strategies hold for the cases where $\gamma_1 = 0$ and $\gamma_2 = 0$. We now move on to the scenario where $\gamma_i > 0 \; \forall \; i$. We start by looking at all the different partitions X and Y of the restaurants. Since the set X and Y must be non-empty we can make six different partitions of three restaurants as shown in Table 3.1. Without loss of any generality, we can consider only the three cases R_1, R_2, and R_3. For these cases we have,

R_1 : In this case we have, $r_2 = r_3 = q_1 = 0, r_1 = 1$ and $q_2 + q_3 = 1$; that imply, $\gamma_1 = \frac{2}{3}$ and $\gamma_2 + \gamma_3 = \frac{1}{3}$.

R_2 : Here, $r_1 = r_3 = q_2 = 0, r_2 = 1$ and $q_1 + q_3 = 1$; which further imply, $\gamma_2 = \frac{2}{3}$ and $\gamma_1 + \gamma_3 = \frac{1}{3}$.

R_3 : In this case, $r_1 = r_2 = q_3 = 0, r_3 = 1$ and $q_1 + q_2 = 1$; and consequently, $\gamma_3 = \frac{2}{3}$ and $\gamma_1 + \gamma_2 = \frac{1}{3}$.

The other three can be achieved from these three by inverting Alice's encoding $0 \to 1$ and $1 \to 0$. Therefore, only those $\mathbb{H}^3(\gamma_1, \gamma_2, \gamma_3)$ are perfectly winnable with a classical mixed strategy where one of the γ_i value is 2/3 or 0 (see Fig. 3.2). In particular, the game $\mathbb{H}^3(1/3)$ is not winnable by a classical strategy.

Set	R_1	R_2	R_3	R_4	R_5	R_6
X	$\{1\}$	$\{2\}$	$\{3\}$	$\{2,3\}$	$\{1,3\}$	$\{1,2\}$
Y	$\{2,3\}$	$\{1,3\}$	$\{1,2\}$	$\{1\}$	$\{2\}$	$\{3\}$

TABLE 3.1 Six different partitioning of three restaurants into two nonempty disjoint sets.

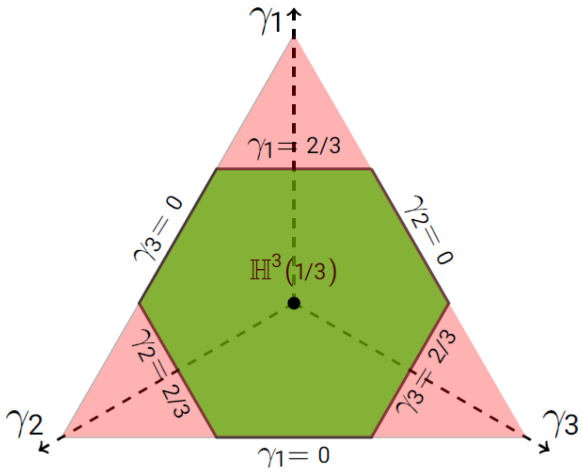


FIG. 3.2 Parameter-space ($\gamma_1 + \gamma_2 + \gamma_3 = 1$ plane) of the games $\mathbb{H}^3(\gamma_1, \gamma_2, \gamma_3)$. Orange shaded regions are the unphysical games as the conditions (h1′) and (h2′) cannot be satisfied for the parameters values chosen from there. The green shaded region (polytope) are all games winnable with correlated strategies with 1-bit of shared randomness. The boundaries of the green polytope, *i.e.* $\gamma_i = 0$ and $2/3$, for $i \in \{1, 2, 3\}$, are the only games winnable with classical mixed strategies.

Even restaurant: unwinnable game

We have already seen that all the games $\mathbb{H}^n(1/n)$ are perfectly winnable with classical mixed strategies whenever n is even. Naturally, the question arises whether all even restaurant games are winnable with such strategies when generic cases are considered. We answer this question negatively by constructing an explicit example of such a game. Note that Eq.(3.10) yields,

$$\sum_{m \in X} \gamma_m = \frac{|Y|}{n}, \quad \& \quad \sum_{m \in Y} \gamma_m = \frac{|X|}{n}. \tag{3.15}$$

Now it becomes easy to come up with an even restaurant game that is impossible to win using a mixed strategy. For instance, consider the 4-restaurant game specified by $\gamma_1 = 2/5$ and $\gamma_2 = \gamma_3 = \gamma_4 = 1/5$. No non-empty disjoint partitioning of the four restaurants satisfies the conditions in Eq.(3.15). Thus the game $\mathbb{H}^4\left(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$ is not winnable with classical mixed strategies.

3.2.3 Games winnable with classical correlated strategies

We have already seen that only a subclass of games $\mathbb{H}^n(\gamma_1, \cdots, \gamma_n)$ can be won with mixed strategies. If Alice and Bob are allowed to share classical shared randomness, then they can follow any correlated strategy and can win any of the game $\mathbb{H}^n(\gamma_1,\cdots,\gamma_n)$. For instance, consider the 3-restaurant case first. Alice and Bob can share a random variable $\lambda \in \{1,2,3\}$ which determines the partitions R_λ that they use in a particular run. Using R_1 they can win all games of the form $(\gamma_1 = \frac{2}{3}, \gamma_2, \gamma_3)$, using R_2 they can win all games of the form $(\gamma_1, \gamma_2 = \frac{2}{3}, \gamma_3)$, and using R_3 they can win all games of the form $(\gamma_1, \gamma_2, \gamma_3 = \frac{2}{3})$. Therefore, using shared randomness they can win all games of the form,

$$r_1\left(\frac{2}{3},a_2,a_3\right)+r_2\left(b_1,\frac{2}{3},b_3\right)+r_3\left(c_1,c_2,\frac{2}{3}\right),$$

where $a_2+a_3=b_1+b_3=c_1+c_2=\frac{1}{3}$ and $r_1+r_2+r_3=1$ and these cover all possible games $\mathbb{H}^3(\gamma_1,\gamma_2,\gamma_3)$.

The above strategy uses $\log 3$ bits of shared randomness. However, a more efficient protocol is possible if we look into the geometry of the game space. The set of all possible games $\mathbb{H}^3(\gamma_1,\gamma_2,\gamma_3)$ forms a polytope embedded in the 2-simplex as shown in Fig.3.2. All games that are winnable through a strategy corresponding to the partition R_1 form the facet $\gamma_1 = \frac{2}{3}$ of the green polytope. Similarly, partitions R_2 and R_3 form facets that correspond to $\gamma_2 = \frac{2}{3}$ and $\gamma_3 = \frac{2}{3}$, respectively. Any game lying within the green polytope can be won by taking a suitable convex mixture of two points from the facets $\gamma_i = 0$ and $\gamma_i = \frac{2}{3}$; $i \in \{1,2,3\}$. Therefore 1-bit of shared randomness along with 1-bit classical channel suffice to construct a perfect strategy for any of the games $\mathbb{H}^3(\gamma_1,\gamma_2,\gamma_3)$.

Moving to the general case, any of the games $\mathbb{H}^n(\gamma_1,\cdots,\gamma_n)$ can be perfectly won with 1-bit of classical communication if sufficient amount of shared randomness is allowed. A trivial upper bound can be obtained from the convex structure of game space. For general n, the game space is a convex polytope embedded in $\mathbb{R}^n$. The extreme points of this polytope are the games specified by the vectors of the form $\left(\frac{n-1}{n},\frac{1}{n},0,\cdots,0\right) \in \mathbb{R}^n$, and its all possible permutations, total $n(n-1)$ in number. All such games can be won with just 1-bit of classical communication without requiring any shared randomness. For instance, while playing the game $\mathbb{H}^n\left(\frac{n-1}{n},\frac{1}{n},0,\cdots,0\right)$, Alice will send '0' if the first restaurant is closed, else she sends '1'. Bob visits the second restaurant when he receives '0', else he visits the first restaurant. It is not hard to see that both the conditions (h′1) and (h′2) are satisfied with this strategy. Since any vector $(\gamma_1,\cdots,\gamma_n) \in \mathbb{R}^n$ specifying the game $\mathbb{H}^n(\gamma_1,\cdots,\gamma_n)$ can always be expressed as a convex mixture of aforesaid $n(n-1)$ extreme points, therefore $\log(n^2-n)$-bit of shared randomness, along with 1-bit of communication, will suffice for winning any of the games

$\mathbb{H}^n(\gamma_1, \cdots, \gamma_n)$. However, this seems an extreme overestimate, and we believe that the games can be won with much less amount of shared randomness. In fact, it would be quite interesting to show that any such game can be won with 1-bit of communication when aided with just 1-bit of shared randomness. The above result can be summarized as the following theorem.

Theorem 1. *Shared randomness can increase the utility of a perfect 1 cbit classical channel in H-FW scenario.*

A game $H^n(\gamma_1, \cdots, \gamma_n)$ winnable with some classical correlated strategy but not winnable with any classical mixed strategy establishes the fact that classical shared randomness can empower utility of a perfect but limited classical communication line in the Holevo & Frenkel-Weiner kind of scenario. Important to note that the single-shot utility of a perfect classical channel gets empowered with shared randomness. This, in a sense, can be thought as a classical version of 'quantum superdense coding' phenomenon where also single-shot communication utility of a perfect quantum channel gets enhanced with preshared entanglement. Of course, there is a crucial difference between these two. In the case of quantum superdense coding channel's utility is quantified through classical mutual information between senders and receivers data which can be increased with preshared entanglement. In the classical scenario, shared randomness cannot increase the mutual information. Therefore in the classical scenario, the usefulness of shared randomness in enhancing the single-shot utility of a perfect channel must be analyzed with some payoff different from classical mutual information. The winning condition of our restaurant games stands for one such payoff. Our restaurant game is only a particular example of tasks that establish such nontrivial communication utility of classical shared randomness. Nonetheless, it motivates further research to explore such novel usefulness of shared randomness in single-shot paradigm. In the following we, however, proceed to analyze the restaurant games with quantum resource.

3.3 Communication advantage of quantum system

When playing the restaurant games in quantum scenario, Alice can send a qubit system to Bob instead of a classical bit. Like classical mixed strategies, no classical shared randomness is allowed between Alice and Bob. In such a case, a quantum strategy can be defined as follows.

Definition 4 (Quantum strategy)**.** *A quantum strategy is an encoding-decoding tuple* $(\mathrm{E}_q, \mathrm{D}_q)$*, where* E_q *is a '*$\log n$*-bit to* 1*-qubit' bijective function and* D_q *is a* *n outcome POVM ,i.e.* $\mathrm{E}_q : i \mapsto \rho_i \in \mathscr{D}(\mathbb{C}^2)$*, with* $i \in \{1, \cdots, n\}$*; and* $\mathrm{D}_q : \{\pi_j \mid \pi_j \geq 0 \ \& \ \sum_{j=1}^{n} \pi_i = \mathbb{I}\}$*, where* $\mathbb{I}$ *is the identity operator on* $\mathbb{C}^2$*.*

Bob makes the decision to visit the restaurants based on his measurement outcomes. Here we assume that the quantum communication line is perfect, *i.e.* the qubit state, Alice intends to send Bob, does not get interrupted with noise. Noisy case analysis we defer to the later section. Interestingly, we will show that there exist quantum strategies that are advantageous over classical mixed strategies.

3.3.1 Special case: $\mathbb{H}^n(1/n)$

Let us first consider the case $n = 3$. For encoding, Alice chooses a symmetric set of pure states lying on an equilateral triangle from a great circle of the Bloch sphere. When the k^{th} restaurant is closed, she sends the state $\rho_k = |\psi_k\rangle\langle\psi_k| = \frac{1}{2}[\mathbb{I} + \hat{n}_k \cdot \vec{\sigma}]$, where $\hat{n}_k$ is the Bloch vector $\left(\sin\frac{2\pi(k-1)}{3}, 0, \cos\frac{2\pi(k-1)}{3}\right)^{\mathrm{T}} \in \mathbb{R}^3$; $k \in \{1,2,3\}$ and T denotes transposition. For decoding, Bob performs the measurement $\mathscr{M}(3) \equiv \left\{\pi_k := \frac{1}{3}[\mathbb{I} - \hat{n}_k \cdot \vec{\sigma}]\right\}_{k=1}^{3}$, and visits the k^{th} restaurant if outcome corresponding to the effect π_k is observed. With this strategy we have $p(i_b|i_c) = \mathrm{Tr}[\rho_i \pi_i] = 0$, $\forall\, i$, ensuring the condition (h1). Furthermore, Bob's probability of visiting m^{th} restaurant turns out to be,

$$
\begin{aligned}
p(m_b) &= \frac{1}{3}\sum_k p(m_b|k_c) = \frac{1}{3}\sum_k \mathrm{Tr}\left[\frac{1}{2}[\mathbb{I} + \hat{n}_k \cdot \vec{\sigma}]\,\frac{1}{3}[\mathbb{I} - \hat{n}_m \cdot \vec{\sigma}]\right] \\
&= \frac{1}{9}\sum_k [1 - \hat{n}_k \cdot \hat{n}_m] = \frac{1}{9}\left[3 - \left(\sum_k \hat{n}_k\right) \cdot \hat{n}_m\right] = \frac{1}{3},
\end{aligned}
$$

which ensures the condition (h2). The strategy can be generalized for any of the games $\mathbb{H}^n(1/n)$ with odd n. Recall that, for even n the game $\mathbb{H}^n(1/n)$ is winnable with classical mixed strategy. For an arbitrary odd n, Alice uses the encoding $\rho_k = \frac{1}{2}[\mathbb{I} + \hat{n}_k \cdot \vec{\sigma}]$, where $\hat{n}_k$'s are symmetrically distributed on the equatorial plane. Bob performs the measurement $\mathscr{M}(n) \equiv \left\{\pi_k := \frac{1}{n}[\mathbb{I} - \hat{n}_k \cdot \vec{\sigma}]\right\}_{k=1}^{n}$ and visits the k^{th} restaurant if outcome corresponding to the effect π_k is observed. A similar calculation as before ensures both the conditions (h1) and (h2).

3.3.2 Three-restaurant general case: $\mathbb{H}^3(\gamma_1, \gamma_2, \gamma_3)$

In this general case, to fulfill the condition (h1$'$), Alice must choose some pure states for encoding. Let she sends the state $|\psi_i\rangle$ to Bob when the i^{th} restaurant is closed. To satisfy the condition (h1$'$) Bob must perform a decoding measurement of the form $\mathscr{M} = \left\{\alpha_i |\psi_i^{\perp}\rangle\langle\psi_i^{\perp}| \mid \alpha_i > 0 \ \& \ \sum_i \alpha_i = 2\right\}_{i=1}^{3}$ and he visits the i^{th} restaurant if the i^{th} effect clicks. To satisfy this requirement the completely mixed

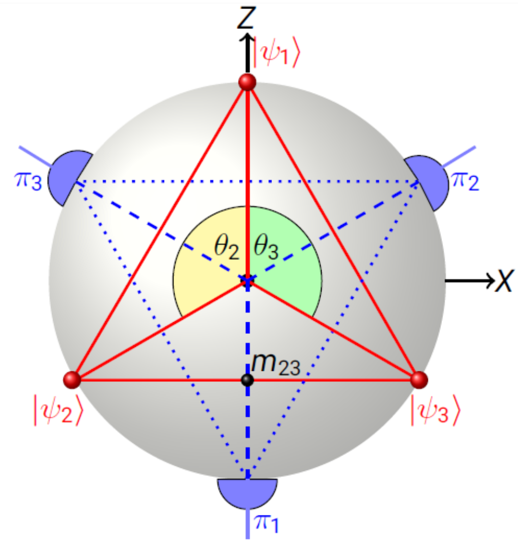


FIG. 3.3 Quantum strategy for $\mathbb{H}^3(\gamma_1, \gamma_2, \gamma_3)$. Alice sends the qubit $|\psi_k\rangle$ when the k^{th} restaurant is closed. Red dots denote Alice's encodings. Blue half-circles denote the rank-one effects corresponding to Bob decoding. Blake dot denotes the mid-point m_{23} of the cord joining the Bloch vectors ψ_2 & ψ_3. Here the strategy is shown for $\gamma_1 = \gamma_2 = \gamma_3 = 1/3$, *i.e.*, for the game $\mathbb{H}^3(1/3)$. For other cases, θ_2 and θ_3 need to be varied accordingly.

state must lie within the triangle formed by Bloch vectors corresponding to the encoding states $\{|\psi_i\rangle\}_{k=1}^3$. Without loss of any generality, Alice can choose her encodings as $\psi_1 = (0,0,1)^{\mathrm{T}}$, $|\psi_2\rangle = (-\sin\theta_2, 0, \cos\theta_2)^{\mathrm{T}}$ and $|\psi_3\rangle = (\sin\theta_3, 0, \cos\theta_3)^{\mathrm{T}}$; where ψ_i is the Bloch vector of the state $|\psi_i\rangle$ and $\theta_2, \theta_3 \in [0, \pi]$[1] are the polar angles for the corresponding Bloch vectors (see Fig.3.3). Accordingly, the requirements of the condition (h1′) read as,

$$\alpha_1 + \alpha_2 + \alpha_3 = 2, \tag{3.16a}$$

$$\alpha_1 + \alpha_2 \cos\theta_2 + \alpha_3 \cos\theta_3 = 0, \tag{3.16b}$$

$$-\alpha_2 \sin\theta_2 + \alpha_3 \sin\theta_3 = 0. \tag{3.16c}$$

[1]Note that $\theta_2 + \theta_3$ cannot be lesser than π, since this configuration would not form a valid measurement.

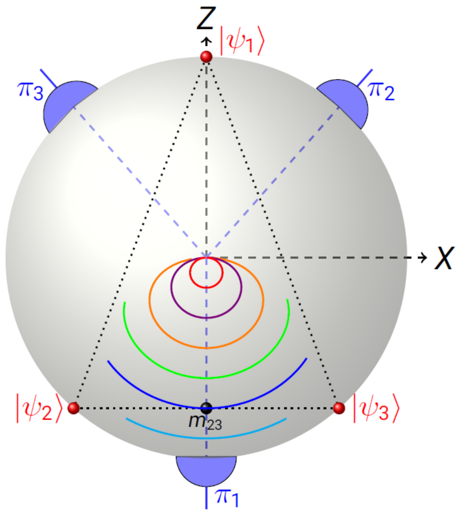


FIG. 3.4 This figure shows locus of the midpoints m_{23} having constant γ_1. Once γ_1 is fixed, m_{23} completely specifies the value of γ_2 and γ_3. Thus it is sufficient to plot constant γ_1 curves. Here, $\gamma_1 = 0.6,\ 0.5,\ 0.4,\ 0.3,\ 0.2, \&\ 0.1$ curves are plotted. The black dotted triangle shows an explicit strategy for the game $\mathbb{H}^3\,(0.5, 0.25, 0.25)$. As the black dot moves on the blue curve we get strategies for the games of the form $\mathbb{H}^3\left(0.5, \frac{p}{2}, \frac{1-p}{2}\right)$, with $p \in [0,1]$. The leftmost point on the blue curve corresponds to the game $\mathbb{H}^3\,(0.5, 0, 0.5)$ while the rightmost point corresponds to $\mathbb{H}^3\,(0.5, 0.5, 0)$.

These further lead to,

$$\alpha_1 = \frac{2\sin(\theta_2+\theta_3)}{\sin(\theta_2+\theta_3)-\sin\theta_2-\sin\theta_3}, \tag{3.17a}$$

$$\alpha_2 = \frac{-2\sin\theta_3}{\sin(\theta_2+\theta_3)-\sin\theta_2-\sin\theta_3}, \tag{3.17b}$$

$$\alpha_3 = \frac{-2\sin\theta_2}{\sin(\theta_2+\theta_3)-\sin\theta_2-\sin\theta_3}, \tag{3.17c}$$

and accordingly, we have,

$$\begin{aligned}\gamma_1 &= \frac{1}{3}(p(1|2)+p(1|3))\\ &= \frac{1}{3}\mathrm{Tr}[\alpha_1|\psi_1^{\perp}\rangle\langle\psi_1^{\perp}|(|\psi_2\rangle\langle\psi_2|+|\psi_3\rangle\langle\psi_3|)]\\ &= \frac{1}{3}\frac{\sin(\theta_2+\theta_3)(2-\cos\theta_2-\cos\theta_3)}{(\sin(\theta_2+\theta_3)-\sin\theta_2-\sin\theta_3)}.\end{aligned} \tag{3.18}$$

The aforesaid encoding can be uniquely specified by specifying the state $|\psi_1\rangle = |0\rangle$ and fixing the midpoint m_{23} of the line joining Bloch vectors of $|\psi_2\rangle$ and $|\psi_3\rangle$ (see

Fig.3.4). This is because for any point within a great circle (except the center) there exists a unique chord having that point as the midpoint of the chord. Thus choosing the encoding state $|\psi_1\rangle = |0\rangle$ when the first restaurant is closed, the complete encoding is specified just by the midpoint m_{23}, where the Bloch vector corresponding to the restaurant 2 lies on the left side. Now, Eq.(3.18) can be used to plot locus of the midpoints m_{23} with constant γ_1. Such a plot is shown in Fig.3.4. We thus conclude that all games $\mathbb{H}^3(\gamma_1,\gamma_2,\gamma_3)$ are perfectly winnable through some quantum strategy. This leads us to the following theorem.

Theorem 2. *In the absence of any preshared correlation utility of 1 qubit communication is higher than 1 bit of classical communication in H-FW scenario.*

3.3.3 Quantum strategy for $\mathbb{H}^4\left(\frac{2}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5}\right)$

In case of any four restaurant game of the form $\mathbb{H}^4\left(\gamma_1,\frac{1-\gamma_1}{3},\frac{1-\gamma_1}{3},\frac{1-\gamma_1}{3}\right)$ with $\gamma_1 > 0$, Alice sends the state $|\psi_i\rangle$ to Bob when the i^{th} restaurant is closed. Without loss of generality, she can choose $|\psi_1\rangle \equiv (0,0,1)^{\mathrm{T}}$, and due to symmetry of the visiting probability ($\gamma_2 = \gamma_3 = \gamma_4$) the encoding of other three state will orient symmetrically on a constant z plane. One can orient $|\psi_2\rangle$ along the x axis on that constant z plane. However, once $|\psi_2\rangle$ is chosen the encodings for $|\psi_3\rangle$ and $|\psi_4\rangle$ are fixed. Let Alice's encodings are

$$
\begin{aligned}
1 &\mapsto |\psi_1\rangle \equiv (0,0,1)^{\mathrm{T}}, \\
2 &\mapsto |\psi_2\rangle \equiv (\sin\theta, 0, \cos\theta)^{\mathrm{T}}, \\
3 &\mapsto |\psi_3\rangle \equiv \left(-\frac{1}{2}\sin\theta, \frac{\sqrt{3}}{2}\sin\theta, \cos\theta\right)^{\mathrm{T}}, \\
4 &\mapsto |\psi_4\rangle \equiv \left(-\frac{1}{2}\sin\theta, -\frac{\sqrt{3}}{2}\sin\theta, \cos\theta\right)^{\mathrm{T}}.
\end{aligned}
$$

To satisfy the condition (h1′), Bob must perform a decoding measurement of the form $\mathscr{M} = \left\{\alpha_i|\psi_i^{\perp}\rangle\langle\psi_i^{\perp}| \mid \alpha_i > 0 \;\&\; \sum_i \alpha_i = 2\right\}_{i=1}^{4}$ and he visits the i^{th} restaurant when i^{th} effect clicks. With this encodings and decoding, the condition (h2′)

further demands

$$\alpha_1+\alpha_2+\alpha_3+\alpha_4 = 2, \tag{3.19a}$$
$$(2\alpha_2-\alpha_3-\alpha_4)\times\sin\theta = 0, \tag{3.19b}$$
$$(\alpha_3-\alpha_4)\times\sin\theta = 0, \tag{3.19c}$$
$$\alpha_1+(\alpha_2+\alpha_3+\alpha_4)\times\cos\theta = 0, \tag{3.19d}$$
$$(1-\cos\theta)\times\alpha_1 = \frac{8}{3}\gamma_1. \tag{3.19e}$$

Solving this set of equations for $\gamma_1=\frac{2}{5}$ we obtain $\alpha_1=\frac{16}{23},\alpha_2=\alpha_3=\alpha_4=\frac{10}{23}$ and $\cos\theta=-\frac{8}{15}$, which completely specify the perfect strategy with qubit communication.

This shows only a particular example where qubit communication yields the perfect strategy for the four-restaurant case. It might be interesting to analyze the general four-restaurant scenario $\mathbb{H}^4(\gamma_1,\gamma_2,\gamma_3,\gamma_4)$. In fact, it would be worth interesting to classify the games $\mathbb{H}^4(\gamma_1,\cdots,\gamma_n)$ that allow perfect qubit strategies. This question we left for future research. In the next, we rather move to analyze the source of the obtained advantage from a more foundational perspective.

3.4 Origin of the advantage

Quantum advantages are hard to find and even harder to prove. In this section, we will analyze which particular non-classical features underlies to the aforesaid communication advantage we have obtained. We will also analyze whether similar advantages are possible or not with some hypothetical non-classical toy systems.

3.4.1 Two no-go results in quantum scenario

In search of the non-classical features of quantum theory that exhibit the above advantage in the communication scenario, here we present two important *no-go* results.

Proposition 1. *A qudit is no better than a c-dit, in H-FW scenario, if Alice encodes the inputs in d-dimensonal commuting density operators.*

Proof. Let us consider that Alice uses n commuting density operator, *i.e.* $\{\rho_1,\rho_2,\cdots,\rho_n\}$, where $\rho_i\in\mathscr{D}(\mathbb{C}^d)$ for her encoding. let the common eigenstate of the commuting density operators be $\{|\phi_1\rangle,|\phi_2\rangle,\cdots,|\phi_d\rangle\}$, then the encoded density operators can be express as $\rho_i=\sum_{j=1}^{d}C_{ij}|\phi_j\rangle\langle\phi_j|$. After receiving the encoded qudit, the receiver performs an n-outcome positive operator valued measure (POVM) $\mathscr{M} \equiv \{E_k : \sum_k E_k = \mathbb{I}\}_{k-1}^n$ to decide among n-possibilities. The probability of making k^{th} decision by the receiver is $p(E_k|i) = \text{Tr}(E_k\rho_i) = \sum_{j=1}^d C_{ij}\text{Tr}(E_k|\phi_j\rangle\langle\phi_j|) = \sum_{j=1}^d C_{ij}\ p(E_k|\phi_j)$, when he receives the state ρ_i; $i \in \{0,1,\cdots,n\}$. In the analogous classical scenario, depending upon the input received from Refree, Alice chooses one of the d-faced coin C_i with the bias C_{ij} where $i \in \{1,2,\cdots,n\}$; $j \in \{1,2,\cdots,d\}$. Depending on the outcome of the coin flip C_i, Alice will send Bob a classical digit ranging from 0 to d, instead of the qudit $\{\rho_1,\rho_2,\cdots,\rho_d\}$. Bob will then choose one of the n-faced coins $\tilde{C}_i$ based on the digit he received from Alice. Each of these coins has a bias $\{p(E_1|\phi_i),\cdots,p(E_n|\phi_i)\}$, where $i \in \{1,2,\cdots,n\}$. Therefore, the probability generated by commuting qudits encoding can always be simulated by a d-bit classical communication. This completes the proof. □

The above result proves the necessity of non-commuting encoding to exhibit the advantage of qudit communication over the d-bit classical channel in the restaurant game $\mathbb{H}^n(\gamma_1,\cdots,\gamma_n)$. Our next result deals with the decoding part at Bob's end.

Proposition 2. *An advantage of a qudit over its classical counterpart in the H-FW scenario requires the non-commutativity of the measurement operators at the receiver's end for decoding.*

Proof. Let the sender uses one among n-qudits $\{|\psi_1\rangle,|\psi_2\rangle,\cdots,|\psi_n\rangle\}$ to encode her messages and sends the encoded state to the receiver. Consider that Bob performs a n-outcome positive operator valued measure (POVM) $\mathscr{M} \equiv \{E_k : \sum_k E_k = \mathbb{I}\}_{k-1}^n$ to decide among n-possibilities, where all the measurement operators *i.e.* E_k commute. So the measurement operators can be expressed as $E_k = \sum_{i=1}^d C_{ki}|\phi_i\rangle\langle\phi_i|$, where $|\phi_i\rangle$ are the common eigenstate of the measurement operators E_k. The encoding states $|\psi_j\rangle$ can be expressed in this eigen basis *i.e.* $|\psi_j\rangle = \sum_{i=1}^d \alpha_{ji}|\phi_i\rangle$. In the classical counterpart, the sender can replace the quantum state $|\psi_j\rangle$ by a source S_j generating classical random variables $\{0,1,\cdots,n\}$ with probabilities $\{|\alpha_{j1}|^2,|\alpha_{j2}|^2,\cdots,|\alpha_{jd}|^2\}$. After receiving the classical random variable Bob will choose one of the n-faced coins C_i with C_{ki} bias and the receiver can simulate the quantum probabilities obtained through commuting decoding. This proves the claim. □

The quantum advantage in our restaurant game, therefore, must invoke 'quantum interference' in the form of quantum superposition at encoding and noncommutivity at decoding steps. This is quite similar the communication advantage as obtained in [26, 51, 60] within the W-ANTV scenario. Here, non-commutative measurements, more specifically measurements that are non-jointly measurable, play a crucial role in the decoding step at the receiver's end. Within the H-FW scenario, the entanglement assisted advantage of a perfect classical channel as reported in [49] once again uses non-compatible measurements at the decoding step.

3.4.2 Is the advantage strictly quantum?

This question is quite important from a foundational perspective. Recall that while the Seminal result of J. S. Bell establishes 'nonlocal' behavior of quantum correlations [15], later Popescu and Rohrlich report such correlations which are beyond quantum in nature [37]. On the other hand, communication advantages of a qubit over its classical counterpart in the W-ANTV kind of scenario can also be obtained with a hypothetical non-classical, in fact with better than quantum success [52]. In a similar way, the advantage reported in the present paper can also be obtained with hypothetical non-classical systems known as polygon theories.

Polygon theories

This class of models can be specified by the tuple $\mathscr{P}_{ly}(n) \equiv (\Omega(n), \mathscr{E}(n))$ [42], where $\Omega(n)$ is the state space and $\mathscr{E}(n)$ is the effect space. For a fixed n, $\Omega(n)$ is the convex hull of n pure states $\{\omega_i\}_{i=1}^{n}$, where $\omega_i := (r_n \cos(2\pi i/n), r_n \sin(2\pi i/n), 1)^{\mathrm{T}} \in \mathbb{R}^3$ with $r_n := \sqrt{\sec(\pi/n)}$. The set $\mathscr{E}(n)$ is the convex hull of the zero effect $z := (0,0,0)^{\mathrm{T}}$, the unit effect $u := (0,0,1)^{\mathrm{T}}$ and the extremal effects $\{e_i, \bar{e}_i = u - e_i\}_{i=1}^{n}$ where,

Odd gon	Even gon
$e_i := \frac{1}{1+r_n^2}\begin{pmatrix} r_n \cos\frac{2\pi i}{n} \\ r_n \sin\frac{2\pi i}{n} \\ 1 \end{pmatrix}$	$e_i := \frac{1}{2}\begin{pmatrix} r_n \cos\frac{(2i-1)\pi}{n} \\ r_n \sin\frac{(2i-1)\pi}{n} \\ 1 \end{pmatrix}$

Although these are just toy models, in the recent past several interesting results have been reported exploring different non-classical aspects of these models which in turn provide better understandings of the Hilbert space structure of quantum theory [88–93]. Here we study these models to explore their communication utility while playing the restaurant games.

Playing $\mathbb{H}^n(1/n)$ with Polygons

For any n, Alice choose the system $\mathscr{P}_{ly}(2n)$ and sends the state $s_i = \frac{1}{2}(\omega_{2i-1} + \omega_{2i})$ to Bob when i^{th} restaurant is closed. Note that, unlike the quantum case, here mixed states may be used for encoding. However, the states lies at the boundary of the polygons. For decoding, Bob performs an n-outcome measurements $\mathscr{M}(n) \equiv \{\frac{2}{n}\bar{e}_{2k}\}_{k=1}^{n}$ and visits k^{th} restaurant if outcome corresponding to the effect $\frac{2}{n}\bar{e}_{2k}$ is observed. The fact, $\bar{e}_{2i} \cdot \omega_{2i-1} = 0$ and $\bar{e}_{2i} \cdot \omega_{2i} = 0$ yield $p(i_b|i_c) = \frac{2}{n}\bar{e}_{2i} \cdot s_i = 0$. Furthermore, $p(m_b) = \frac{1}{n}\sum_{k=1}^{n} p(m_b|k_c) = \frac{1}{n}\sum_{k=1}^{n} \frac{2}{n}\bar{e}_{2m} \cdot s_k = \frac{2}{n^2}\bar{e}_{2m} \cdot \sum_{k=1}^{n} s_k = \frac{2}{n^2}\bar{e}_{2m} \cdot nu = \frac{1}{n}$. Thus both the conditions (h1) and (h2) are satisfied (see Fig.3.5).

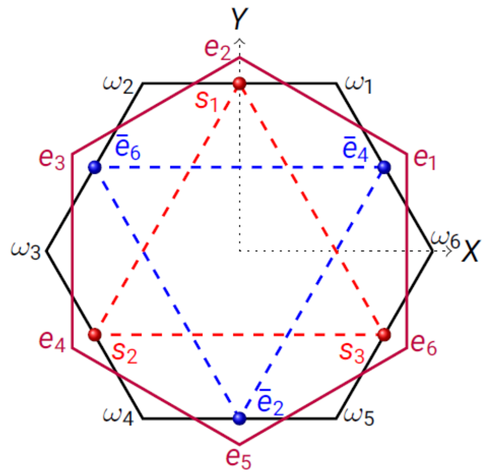


FIG. 3.5 Strategy for the $\mathbb{H}^3(1/3)$ game with communication of a single $\mathscr{P}_{ly}(6)$ system from Alice to Bob (without shared randomness). Red dots denote Alice's encodings. Blue dots are the effects proportional to Bob's decoding measurement.

Note that, so far all the advantageous strategies discussed with qubit and polygonal models does not invoke any local randomness at the decoding step by Bob. However, using local randomness at Bob's end it can be further argued that the game $\mathbb{H}^3(1/3)$ can also be perfectly won with $\mathscr{P}_{ly}(4)$ system. For that, Alice uses the encoding $\mathbb{E} \equiv \{1 \mapsto \omega_1,\ 2 \mapsto \omega_4,\ 3 \mapsto s := \frac{1}{2}(\omega_2 + \omega_3)\}$. For decoding, Bob performs the measurement $\mathscr{M} \equiv \{\frac{1}{2}e_1, \frac{1}{2}e_2, \frac{1}{2}e_3, \frac{1}{2}e_4\}$ which is a convex mixture (use of local randomness at decoding step) of the measurements $\mathscr{M}_1 \equiv \{e_1, e_3\}$ and $\mathscr{M}_2 \equiv \{e_2, e_4\}$ (see Fig.3.6). Bob visits the restaurant 3, 2 and 1, respectively when the outcome corresponding to the effect $e_1/2$, $e_2/2$, and $e_4/2$ clicks. For the effect $e_3/2$ he visits restaurant 1 and 2 with equal probability. It is a straightforward calculation to check that both the conditions (h1) and (h2) are satisfied with this strategy. In fact, it turns out that any $\mathbb{H}^3(\gamma_1.\gamma_2, \gamma_3)$ game can be perfectly won with the $\mathscr{P}_{ly}(4)$ system.

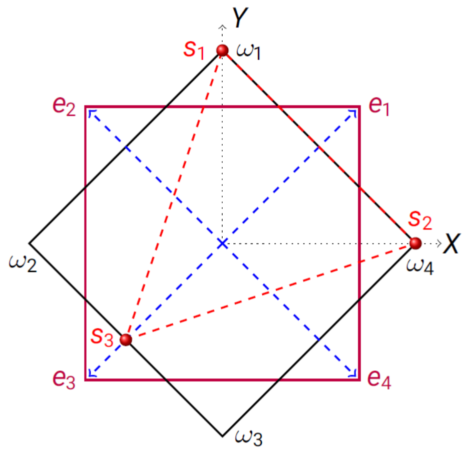


FIG. 3.6 Strategy for the $\mathbb{H}^3(1/3)$ game with communication of single $\mathscr{P}_{ly}(4)$ system from Alice to Bob (without shared randomness). Red dots denote Alice's encodings. Blue dashed lines are the effects proportional to Bob's decoding measurement.

At this point, the work by Massar & Patra is worth mentioning [88]. They have shown that the classical capacity (in Holevo sense *i.e.*, in the asymptotic limit) of n-gon system is 1 bit for even n, whereas it is larger than 1 bit for odd n. In other words, all odd-gons have more communication utility than a single classical bit. On the other hand, the authors in [53] have reported the super-quantum communication utility of polygon models within the W-ANTV communication scenario. More recently, the super-quantum communication utility of $n = 4$ system (square bit) has been established within H-FW scenario by considering suitable bipartite composition of this system [94]. Our restaurant games, however, establish super-classical communication utility of all even gon models in H-FW scenario from single copy consideration only. It might be interesting to compare the communication utility of a qubit with polygon models while playing the restaurant games. We defer this issue to the next section.

3.5 Greater shared randomness, greater utility

So far we have seen that the games $\mathbb{H}^n(\gamma_1, \cdots, \gamma_n)$ proposed above have several novel implications. On the one hand, they establish the important role of classical shared randomness in enhancing the communication utility of a perfect classical channel, on the other hand, they also establish the superiority of qubit (as well as polygon systems) communication over a 1-bit classical channel in H-FW communication scenario. In Section 3.2.3 it has also been shown that an arbitrary $\mathbb{H}^3(\gamma_1, \gamma_2, \gamma_3)$ game can always be perfectly won when 1-bit of classical communication is assisted with 1-bit of classical shared randomness. Here we propose a class of games to show that classical communication lines may need more assistance of shared randomness for perfect winning.

Strict restaurant game: The game denoted as $\mathbb{H}^n[1/(n-1)]$[2] can be seen as a stricter version of $\mathbb{H}^n(1/n)$. Here the conditions (h1) and (h2) are modified as follows

(h_s1) Bob never visits a closed restaurant, *i.e.* $p(i_b|i_c) = 0,\ \forall\ i \in \{1,\cdots,n\}$.

(h_s2) Whenever a certain restaurant is closed all other restaurants must be visited with equal probability, *i.e.* $p(i_b|j_c \neq i) = \frac{1}{n-1},\ \forall\ i,j \in \{1,\cdots,n\}$.

The visit matrix in Eq.(3.1) with this stricter winning conditions can be written compactly as

$$p(i_b|j_c) = \frac{1}{n-1}(1-\delta_{ij}), \tag{3.20}$$

where δ_{ij} denotes the Kronecker delta. Note that, the condition (h_s1) is exactly same as the condition (h1). On the other hand, condition (h_s2) always implies condition (h2), but the converse is not the case in general, which makes this game strict.

3.5.1 Unwinnability of $\mathbb{H}^4[1/3]$ with 1-bit communication +1-bit shared randomness

For the game $\mathbb{H}^4[1/3]$ the winning conditions read as,

$$p(i_b|j_c) = \frac{1}{3}(1-\delta_{ij}); \quad i,j \in \{1,\cdots,4\}. \tag{3.21}$$

As a physical source of classical shared randomness we can consider that Alice and Bob share a two-qubit classically correlated state $\rho_{AB} = \lambda\,|00\rangle_{AB}\langle 00| + (1-\lambda)\,|11\rangle_{AB}\langle 11|$, with $\lambda \in [0,1]$. Outcomes $s \in \{0,1\}$ of their σ_z measurement are correlated in this state. They have 1-bit of shared randomness whenever $\lambda = 1/2$, whereas $\lambda \in \{0,1\}$ implies no shared randomness.

In assistant with 1-bit of shared randomness, the most general strategy that Alice and Bob can follow is a convex mixture of two mixed strategies that can be implemented as the following steps.

(S-1) Depending on the outcome $s \in \{0,1\}$ of the measurement σ_z on her part of the state ρ_{AB} and based on the information about the closed restaurant $k \in$

[2]Please note that here we use 'square bracket' to denote the stricter version game $\mathbb{H}^n[\star]$ to make the distinction from its non-stricter version.

$\{1,\cdots,4\}$, Alice tosses a 2 sided biased coin having the outcomes $\{0,1\}$. The outcome probabilities of the coins are given by $P_k^{(s)}(0) = \alpha_k^{(s)}$ and $P_k^{(s)}(1) = 1-\alpha_k^{(s)}$.

(S-2) Alice communicates the outcome of her coin toss to Bob through the 1-bit classical channel.

(S-3) Depending on the outcome $s \in \{0,1\}$ of σ_z measurement on his part of the state ρ_{AB}, Bob prepares two 4 sided coins with outcomes $\{1,\cdots,4\}$ having the outcome probabilities $\vec{r}^{(s)} = (r_1^{(s)},\cdots,r_4^{(s)})$ and $\vec{q}^{(s)} = (q_1^{(s)},\cdots,q_4^{(s)})$, respectively. Upon receiving 0 from Alice, he tosses the $\vec{r}^{(s)}$ coin and visits the i^{th} restaurant if i^{th} outcome occurs. He follows a similar strategy with the $\vec{q}^{(s)}$ coin if 1 is received from Alice.

With this strategy, the conditional probability $p(m_b|k_c)$ that Bob visits m^{th} restaurant provided the k^{th} restaurant is closed, turns out to be,

$$\begin{aligned} p(m_b|k_c) =& \lambda \left[\alpha_k^{(0)} \times r_m^{(0)} + \left(1-\alpha_k^{(0)}\right) \times q_m^{(0)}\right] + \\ & (1-\lambda) \left[\alpha_k^{(1)} \times r_m^{(1)} + \left(1-\alpha_k^{(1)}\right) \times q_m^{(1)}\right]. \end{aligned} \tag{3.22}$$

The first condition (h_s1) demands $p(i_b|i_c) = 0,\ \forall\, i \in \{1,\cdots,4\}$, which consequently implies,

$$\begin{aligned} \forall\, i,\ \ \lambda & \left[\alpha_i^{(0)} \times r_i^{(0)} + \left(1-\alpha_i^{(0)}\right) \times q_i^{(0)}\right] + \\ (1-\lambda) & \left[\alpha_i^{(1)} \times r_i^{(1)} + \left(1-\alpha_i^{(1)}\right) \times q_i^{(1)}\right] = 0. \end{aligned} \tag{3.23}$$

It is easy to argue that in absence of shared randomness, *i.e.* when $\lambda \in \{0,1\}$, the goal is impossible to achieve. For $\lambda = 0$ we have $\alpha_i^{(1)} \times r_i^{(1)} + \left(1-\alpha_i^{(1)}\right) \times q_i^{(1)} = 0\ \forall\ i\ \in\ \{1,\cdots 4\}$. This boils down to coming up with a partition X and Y as discussed in Section 3.2.2. However, this scenario can be disregarded immediately. As there are 4 restaurants, no matter which partition X and Y of the restaurants we take, there must exist at-least 2 restaurants in either X or Y. Now if two restaurants m and k belong to the same partition we must have $p(m_b|k_c) = 0$; and hence violates the winning condition (h_s2). A similar argument holds for the case $\lambda = 1$. We now go on to show that even with the assistance of shared randomness, *i.e.* for $\lambda \notin \{0,1\}$, the goal is impossible to achieve. Since

$\lambda \in (0,1)$, to satisfy Eq.(3.23) we must have

$$\alpha_i^{(0)} \times r_i^{(0)} = 0, \quad \left(1-\alpha_i^{(0)}\right) \times q_i^{(0)} = 0,$$
$$\alpha_i^{(1)} \times r_i^{(1)} = 0, \quad \left(1-\alpha_i^{(1)}\right) \times q_i^{(1)} = 0,$$
$$\forall\, i\, \in \{1,2,\cdots 4\}.$$

All four equations have products of two terms on the left side, and for each pair, at least one term must be zero. This in turn leads to several possible ways to satisfy the equations. The case where the left term of a particular equation is 0 will be denoted by '0', whereas '1' will indicate the right term is 0. So each restaurant must be assigned a four bit string indicating which of the terms in these four equations are 0. For instance, lets say we assign a string '0110' to the i^{th} restaurant. The first bit of this string implies $\alpha_i^{(0)} = 0$, second implies $q_i^{(0)} = 0$, and similarly third and fourth imply $r_i^{(1)} = \left(1-\alpha_i^{(1)}\right) = 0$. It becomes immediate that the possibilities corresponding to the set of strings $\{0000, 0001, 0010, 0011, 0100, 1000, 1100\}$ are not allowed for any $i \in \{1,\cdots,4\}$. For all these cases we have $\alpha_i^{(s)} = 1-\alpha_i^{(s)} = 0$ for at least one $s \in \{0,1\}$, which is impossible. Furthermore, the possibility corresponding to the string '1111' is also not allowed for any i. In this case, although $p(i_b|i_c) = 0$ and hence the condition (h$_s$1) is satisfied, it turns out that $p(i_b|j_c \neq i) = 0$ violating the condition (h$_s$2). Thus for any of the restaurants, the allowed possibilities are

$$\left\{\begin{matrix} S_1 = 0111, & S_2 = 1011, & S_3 = 1101, & S_4 = 1110, \\ S_5 = 0101, & S_6 = 0110, & S_7 = 1001, & S_8 = 1010 \end{matrix}\right\}.$$

We will use the notation $i^{(S_d)}$ to denote that i^{th} restaurant is assigned the string S_d. Let the string $S_1 = 0111$ is assigned to the i^{th} restaurant, then using Eq.(3.22) we obtain $p(i_b^{(S_1)}|j_c) = \lambda[\alpha_j^{(0)} \times r_i^{(0)}]$. If the same string ($S_1$) is assigned to the j^{th} restaurant for $j \neq i$, then we must have $\alpha_j^{(0)} = 0$. Consequently we have $p(i_b^{(S_1)}|j_c^{(S_1)}) = 0$, which violates the condition (h$_s$2). Thus two different restaurants cannot be assigned the same string, confining us to select 4 different strings for 4 different restaurants. It is important to note that for $i \neq j$, $p(i_b^{(S_5)}|j_c^{(S_1)})$ need not be 0, although $p(i_b^{(S_1)}|j_c^{(S_5)}) = 0$. We say that S_1 is compatible with S_5, but S_5 is incompatible with S_1 as it violates (h$_s$2). The list of compatible assignments of the strings are listed in Table 3.2. To satisfy the condition $p(m_b|k_c) = 1/3\ \forall\ (m \neq k)$

String (S_i)	Strings compatible with S_i
S_1	$S_2,\ S_3,\ S_4,\ S_7,\ S_8$
S_2	$S_1,\ S_3,\ S_4,\ S_5,\ S_6$
S_3	$S_1,\ S_2,\ S_4,\ S_6,\ S_8$
S_4	$S_1,\ S_2,\ S_3,\ S_5,\ S_7$
S_5	$S_1,\ S_2,\ S_3,\ S_4,\ S_6,\ S_7,\ S_8$
S_6	$S_1,\ S_2,\ S_3,\ S_4,\ S_5,\ S_7,\ S_8$
S_7	$S_1,\ S_2,\ S_3,\ S_4,\ S_5,\ S_6\ ,\ S_8$
S_8	$S_1,\ S_2,\ S_3,\ S_4,\ S_5,\ S_6,\ S_7$

TABLE 3.2 On the right column we list all the strings that are compatible with a given string S_k; $k \in \{1,\cdots,8\}$. Although the string S_1 is compatible with the string S_5 (see 5^{th} row), S_5 is not compatible with S_1 (see 1^{st} row).

we must select 4 strings that are compatible with each other. From Table 3.2 we only have two possible ways of selecting the strings: (i) $\equiv \{S_1,S_2,S_3,S_4\}$ or (ii) $\equiv \{S_5,S_6,S_7,S_8\}$. We are left to show that none of these two cases yields any consistent solutions for all the variables $\left\{\lambda,\alpha_k^{(s)},\vec{r}^{(s)},\vec{q}^{(s)}\right\}$.

(i) $\{S_1,S_2,S_3,S_4\}$: Without loss of any generality we can choose the string S_i for i^{th} restaurant. Then using Eq.(3.21) & Eq.(3.22) we obtain

$$p(1|2) = \lambda r_1^{(0)} = \frac{1}{3}, \quad p(1|3) = \lambda \alpha_3^{(0)} r_1^{(0)} = \frac{1}{3},$$
$$p(2|1) = \lambda q_2^{(0)} = \frac{1}{3}, \quad p(2|3) = \lambda(1-\alpha_3^{(0)}) q_2^{(0)} = \frac{1}{3}.$$

A consistent solution does not exist for these equations.

(ii) $\{S_5,S_6,S_7,S_8\}$: Once again without loss of an generality we can choose the string S_{i+4} for i^{th} restaurant. Once agin Eq.(3.21) & Eq.(3.22) yield

$$p(1|2) = (1-\lambda) r_1^{(1)} = \frac{1}{3}, \quad p(1|3) = \lambda r_1^{(0)} = \frac{1}{3},$$
$$p(1|4) = \lambda r_1^{(0)} + (1-\lambda) r_1^{(1)} = \frac{1}{3}.$$

No consistent solution exists in this case too. Thus the game $\mathbb{H}^4[1/3]$ is not possible to win with 1-bit of classical communication even when the communication channel is assisted with 1-bit of classical shared randomness. Naturally, the question arises whether this game is winnable with 1-bit of classical communication if more shared randomness is supplied.

Theorem 3. *In the H-FW scenario the communication utility of 1 cbit channel is increased as the amount of assisting shared randomness is increased from 1 bit to* $\log 3$ *bit.*

Proof. Although the game $\mathbb{H}^4[1/3]$ cannot be won with 1-bit+1-SR resource, here we will show that the goal can be perfectly achieved through a 1-bit classical channel if more shared randomness are provided as assistance. The strategy is described below.

Alice divides the restaurants into two disjoint partitions X and Y, such that $|X| = |Y| = 2$. Alice sends 0 to Bob whenever a restaurant in X is closed, otherwise, she sends 1. Bob visits the restaurants within the set Y (X) with uniform probability if he receives 0 (1) from Alice. Clearly, the condition ($h_s 1$) is satisfied as Bob never visits a closed restaurant. Consider the following three partitionings X and Y.

C-1: $X = \{1,2\}$ & $Y = \{3,4\}$,

C-2: $X = \{1,3\}$ & $Y = \{2,4\}$,

C-3: $X = \{1,4\}$ & $Y = \{2,3\}$.

These correspond to three different strategies leading to three different visit matrices

$$\mathbb{V}_1 = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}, \mathbb{V}_2 = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}, \mathbb{V}_3 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

An equal mixture of these three strategies, which requires Alice and Bob to share $\log 3$-bit of shared randomness, yields the resulting visit matrix

$$\mathbb{V} = \begin{pmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 \end{pmatrix},$$

which satisfies both the conditions ($h_s 1$) and ($h_s 2$).

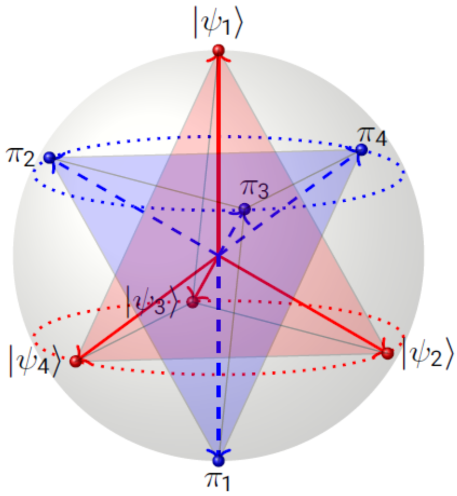


FIG. 3.7 Alice's encodings (red dots) form a symmetric tetrahedron inscribed within the Bloch sphere. For decoding Bob performs SIC-POVM corresponding to the inverted tetrahedron (blue dots) of Alice's encoding tetrahedron.

$\mathbb{H}^{\mathbf{n}}[\mathbf{1}/(\mathbf{n}-\mathbf{1})]$: The above protocol can be generalized for $\mathbb{H}^n[1/(n-1)]$ game with 1-bit of classical communication assisted with $\log(n-1)$-bit of SR. Alice sends 0 and 1 to direct Bob to visit k-th and $(k+1)$-th restaurant respectively, where $k \in \{1,2,\cdots,(n-1)\}$. The value of the k will be identified by the outcomes $\{1,2,\cdots,(n-1)\}$ of the SR, so it requires $\log(n-1)$-bit of SR. Whenever the m-th restaurant ($m \in \{2,3,\cdots,(n-1)\}$) is closed, Alice communicates 0 and 1 respectively for every $k \in \{1,\cdots,(m-1)\}$-th and $k \in \{m,\cdots,(n-1)\}$-th outcomes of the SR. On the other hand, if the 1-st or, n-th restaurant is closed, then Alice will communicate 0 and 1 respectively, independent of the SR outcomes. It is easy to verify that the strategy satisfies both the required conditions. □

Remark 1. *At this point it is important to note that a $\mathbb{H}^n[1/(n-1)]$ game cannot be won with a qubit communication alone, whenever $n \geq 5$. This establishes the 'order of merit' Q$\prec_{inst}$C+SR as listed in Table 3.3.*

3.5.2 Perfect winnability of $\mathbb{H}^4[1/3]$ with qubit strategy

Although the game $\mathbb{H}^4[1/3]$ cannot be won with 1-bit communication +1-bit shared randomness, the goal can be achieved through a qubit communication without any assistance of classical shared randomness. Alice sends the qubit state $|\psi_i\rangle$ when the i^{th} restaurant is closed. The Bloch vectors corresponding the encoded states are respectively

$$1 \mapsto |\psi_1\rangle \equiv (0,0,1)^{\mathrm{T}},$$
$$2 \mapsto |\psi_2\rangle \equiv \left(\frac{2\sqrt{2}}{3},0,\frac{-1}{3}\right)^{\mathrm{T}},$$
$$3 \mapsto |\psi_3\rangle \equiv \left(\frac{2\sqrt{2}}{3}\cos\frac{2\pi}{3},\frac{2\sqrt{2}}{3}\sin\frac{2\pi}{3},\frac{-1}{3}\right)^{\mathrm{T}},$$
$$4 \mapsto |\psi_4\rangle \equiv \left(\frac{2\sqrt{2}}{3}\cos\frac{4\pi}{3},\frac{2\sqrt{2}}{3}\sin\frac{4\pi}{3},\frac{-1}{3}\right)^{\mathrm{T}}.$$

The encodings form a symmetric tetrahedron within the Bloch sphere (see Fig.3.7). For decoding, Bob performs the measurement $\mathscr{M}(4) \equiv \left\{\pi_k := \frac{1}{2}|\psi_k^{\perp}\rangle\langle\psi_k^{\perp}|\right\}_{k=1}^{4}$ and visits the k^{th} restaurant if outcome corresponding to the effect π_k is observed. Note that, the decoding corresponds to the SIC-POVM described by the tetrahedron inverted to the encoding tetrahedron. The fact, $\mathrm{Tr}\left(|\psi_i\rangle\langle\psi_i|\,\pi_k\right) = \frac{1}{3}(1-\delta_{ik})$ ensures perfect winning of the game $\mathbb{H}^4[1/3]$. For the comparison of resources we will use $\mathscr{R}_1 \prec_{inst} \mathscr{R}_2$ to imply that there is some instance where $\mathscr{R}_2$ is strictly better than $\mathscr{R}_1$, while in a different instance their utilities might be in reverse order. So by considering the $\mathbb{H}^4[1/3]$ game we conclude the following theorem.

Theorem 4. *There exist a task in the H-FW scenario where the utility of 1 qubit is higher than the utility of 1 cbit with the assistance of 1 bit of shared randomness i.e.* $C + 1SR \prec_{inst} Q$.

3.5.3 Unwinnability of $\mathbb{H}^4[1/3]$ with Polygons

To ensure that condition (h_s1) with a polygon system, Alice's must choose her encodings $\{\omega_i\}_{i=1}^4$ from the boundary of the polygon. Otherwise, no effect in a polygon model will yield zero probability for a given encoding state. For an e_i if we move towards the effect $\tilde{e}_i$, with e_i and $\tilde{e}_i$ forming a measurement, the probability of observing the effect e_i decreases gradually. For the game $\mathbb{H}^4[1/3]$, the decoding strategy of Bob must be a 4 outcome measurement $\mathscr{M}(4) \equiv \left\{(\alpha_i, e_i) \mid \sum_{i=1}^4 \alpha_i e_i = u\right\}$ such that Bob visits i^{th} restaurant when the outcome corresponding to the effect e_i is observed. For Alice's encoding $\{\omega_j\}_{j=1}^4$ the winning condition of $\mathbb{H}^4[1/3]$ demands $\alpha_i e_i^{\mathrm{T}} \omega_j = \left(1-\delta_{ij}\right)/3$

For any effect e_i there exists an state ω_j such that $e_i^{\mathrm{T}}\omega_j \leq \frac{1}{3} < e_i^{\mathrm{T}}\omega_{j+1}$. Let the first encoding state is $p\omega_j + (1-p)\omega_{j+1}$, where $p \in [0,1)$. Symmetry of the state spaces imply that probability of observing e_i on any state ω_j is same as that on the state ω_j^*, where ω_j^* is the mirror image of ω_j about the line joining

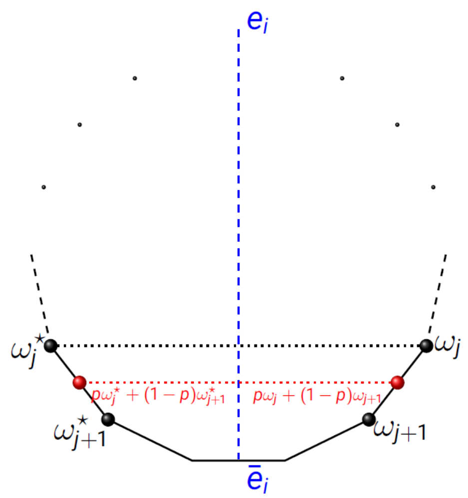


FIG. 3.8 For any polygon theory the probability of observing an effect e_i on a state ω_j keep decreasing as the state moves from e_i towards $\tilde{e}_i$. Since e_i and $\bar{e}_i$ form a measurement, *i.e.*, $e_i + \bar{e}_i = u$, the mirror image ω_j^* of any state ω_j about the line passing through e_i and $\tilde{e}_i$ will give the same probability of occurrence the effect e_i.

e_i and $\tilde{e}_i$, where $e_i + \bar{e}_i = u$ (see Fig. 3.8). Therefore, the second encoding state must be $p\omega_j^* + (1-p)\omega_{j+1}^*$, which is the mirror image of the first encoding state about the line joining e_i and $\bar{e}_i$. As the probability of observing the effect e_i keeps decreasing gradually as the choices of the encoding state move away from e_i to $\bar{e}_i$, there will not be a third encoding state yielding the winning probability for the effect e_i. Thus, a set of 4 distinct encoding states does not exist that satisfy the winning condition. So as a comparison of different communication resources we can establish the following theorem.

Theorem 5. *There exist a task in the H-FW scenario where in the absence of any preshared correlation the utility of 1 qubit is greater than 1 polygon bit i.e. Polygon* $\prec_{inst}$ *Q.*

3.6 Order of merit

Present work along with some recent works put interesting 'order of merit' among different communication resources to compare their utility in the H-FW communication scenario. Following three different ordering relations we will use.

- A resource $\mathscr{R}_1$ will be same as a different resource $\mathscr{R}_2$, denoted as $\mathscr{R}_1 = \mathscr{R}_2$ if for any communication task within H-FW scenario they provide same utility.

- We will denote $\mathscr{R}_1 \preccurlyeq \mathscr{R}_2$ if $\mathscr{R}_2$ is as good as $\mathscr{R}_1$ for any such task and there exist at-least one task where $\mathscr{R}_2$ yield a strictly greater utility than $\mathscr{R}_1$.
- Finally, $\mathscr{R}_1 \prec_{inst} \mathscr{R}_2$ will imply that there is some instance where $\mathscr{R}_2$ is strictly better than $\mathscr{R}_1$, while in a different instance their utilities might be in reverse order.

A few important orderings are listed in Table 3.3.

Order of merit	**Task**
Q $\preccurlyeq$ Q + 1 ebit	PRL **69**, 2881 (1992)
C $\preccurlyeq$ C + 1 ebit	Quantum **6**, 662 (2022)
C+SR=Q+SR	CMP **340**, 563 (2015)
C $\preccurlyeq$ C+SR	$\mathbb{H}^n(\gamma_1,\cdots,\gamma_n)$: Section 3.2.3
C$\preccurlyeq$ Q	$\mathbb{H}^3(\gamma_1,\gamma_2,\gamma_3)$: Section 3.3
C$\preccurlyeq$ $2n$-gon	$\mathbb{H}^n(1/n)$: Section 3.4.2
C+ 1 SR$\prec_{inst}$Q	$\mathbb{H}^4[1/3]$: Section 3.5.2
C+1 SR$\preccurlyeq$ C+$\log 3$-SR	$\mathbb{H}^4[1/3]$: Section 3.5.1
Polygon$\prec_{inst}$ Q	$\mathbb{H}^4[1/3]$: Section 3.5.3
Q$\prec_{inst}$C+SR	$\mathbb{H}^n[1/(n-1)]$, $n \geq 5$: Section 3.5.1
Q+SR$\prec_{inst}$C+1 ebit	Quantum **6**, 662 (2022)

TABLE 3.3 Utility of different resources in H-FW scenario. Here Q denotes a perfect qubit channel and C is a perfect 1-bit classical channel. When we write just 'SR' it means unbounded amount of this resource is allowed.

3.7 Making the quantum advantage noise-robust

Noise is an inevitable enemy of any communication line. In quantum scenario this is even more prominent as a little bit of thermal noise can destroy the encodings. Furthermore, due to measurement impression it might not be possible to perform the intended decoding measurement exactly. So it is quite important to investigate whether the quantum advantage reported above persists under a noisy circumstance. Here we mainly analyse the $\mathbb{H}^3(1/3)$ game. A similar analysis is possible for other cases too. Note that any kind of noise that affects the entire Bloch sphere symmetrically, known as depolarizing (D) noise, does not change the visiting probability $p(m_b)$. Experimentally such a noise can arise when Alice is not able to prepare the ideal quantum preparations $\rho_k = \frac{1}{2}[\mathbb{I} + \hat{n}_k \cdot \vec{\sigma}]$, rather $\hat{n}_k$ takes values uniformly from a cone in the Bloch sphere that subtends solid angle Ω at the origin with axis $\hat{n}_k$ (see Fig.3.9). For such a noise D_ε, specified by depolarizing parameter $\varepsilon \in [0,1]$, a state $\rho_k = \frac{1}{2}[\mathbb{I} + \hat{n}_k.\vec{\sigma}]$ gets modified as

$$\begin{aligned} D_\varepsilon(\rho_k) &:= (1-\varepsilon)\rho_k + \varepsilon\frac{\mathbb{I}}{2} = \frac{1}{2}[\mathbb{I} + (1-\varepsilon)\hat{n}_k.\vec{\sigma}] \\ &= \frac{1}{2}[\mathbb{I} + D_\varepsilon(\hat{n}_k).\vec{\sigma}], \quad D_\varepsilon(\hat{n}_k) := (1-\varepsilon)\hat{n}_k. \end{aligned} \tag{3.24}$$

Similarly, the POVM elements $\pi_k = \frac{1}{3}[\mathbb{I} - \hat{n}_k \cdot \vec{\sigma}]$ get modified to,

$$\begin{aligned} D_\varepsilon(\pi_k) &:= (1-\varepsilon)\pi_k + \varepsilon\frac{\mathbb{I}}{3} = \frac{1}{3}[\mathbb{I} + (1-\varepsilon)\hat{n}_k.\vec{\sigma}] \\ &= \frac{1}{3}[\mathbb{I} - D_\varepsilon(\hat{n}_k).\vec{\sigma}], \quad D_\varepsilon(\hat{n}_k) := (1-\varepsilon)\hat{n}_k. \end{aligned} \tag{3.25}$$

Consider that both Alice's encoding and Bob's decoding get affected with such noises D_{ε_e} and D_{ε_d}, respectively. For the game $\mathbb{H}^3(1/3)$, due to the symmetric nature of the noise, Bob's probability of visiting a restaurant will remain the same as before, *i.e.* $p(m_b) = 1/3$. However, there is a finite probability of visiting a closed restaurant by Bob, with the probability depending on the noise parameters

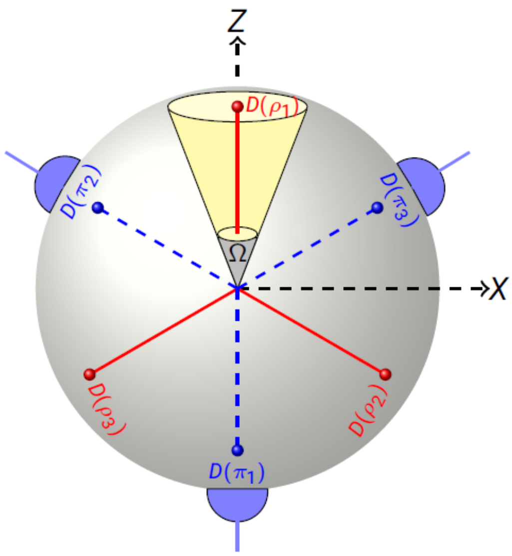


FIG. 3.9 Noise robust quantum protocol for the game $\mathbb{H}^3(1/3)$. Under a depolarizing noise the encoded states get shrieked within the Bloch sphere. Both the encoded states and decoding effects are assumed to undergo such noise.

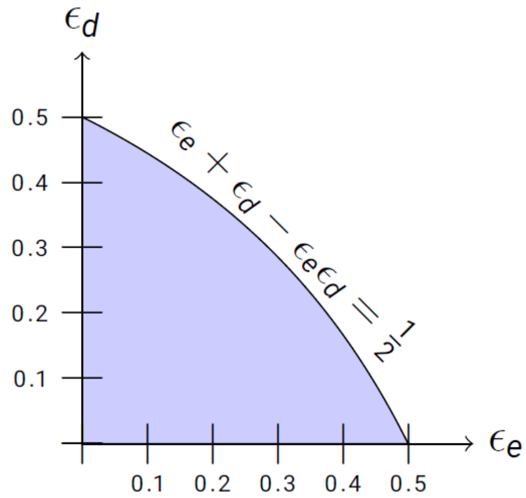


FIG. 3.10 Trade-off in the noise parameters ε_e and ε_d allowing quantum advantage in $\mathbb{H}^3(1/3)$ when both the encodings and decoding experience depolarizing noise.

ε_s and ε_e. The probability of visiting the k^{th} when it is closed is,

$$p(k_b|k_c) = \text{Tr}\left[D_{\varepsilon_{s]e}}(\rho_k)D_{\varepsilon_d}(\pi_k)\right]$$
$$= \frac{1}{3}(\varepsilon_e + \varepsilon_d - \varepsilon_e\varepsilon_d). \tag{3.26}$$

If a classical strategy without shared randomness is used, and we put the conditions that (h_n1) a closed restaurants can be visited with probability less than 1/6 and (h_n2) Each restaurant must be visited with equal probability, then no solution exists; here the sub-index n is used to denote that the perfect conditions is modified for noisy case. In other words, there is no classical mixed strategy without shared randomness that satisfies the conditions:

$$p(m_b) = \frac{1}{3} \;\;\&\;\; p(m_b|m_c) < \frac{1}{6}, \; \forall m \in \{1,2,3\}. \tag{3.27}$$

Therefore, a noisy quantum strategy gives advantage over any classical mixed strategy whenever noise parameters ε_s and ε_e are small enough to obey the condition $(\varepsilon_e + \varepsilon_d - \varepsilon_e\varepsilon_d) < 1/2$ as illustrated in Fig 3.10. At this point, one may wish to analyze a different type of noise in quantum systems and carry out the analysis along the lines that we have done above. For such purposes it is convenient to define a function that captures the error obtained while trying to play our restaurant game. One such function is given by

$$\mathscr{E}(s) = \frac{1}{3}\sum_i \left[p^s(i_b|i_c) + (\gamma_i^s - 1/3)^2\right]; \tag{3.28}$$

where $\gamma_i^s := \frac{1}{3}\sum_{j=1}^3 p^s(j_b|i_c)$ and 's' denotes the strategy followed by Alice and Bob. The collective aim of Alice and Bob is to come up with a strategy that optimize this error. For a perfect strategy (s_p) that satisfies both (h1) and (h2) we have $\mathscr{E}(s_p) = 0$. A general mixed strategy is characterized by 7 parameters – 3 for three different 2-faced coins on Alice's part and 4 for two 3-faced coins on Bob's part. Using Monte Carlo simulation over this 7-parameter space we find that for classical mixed strategies, $\mathscr{E}(s)$ is lower bounded by 0.108. Thus any noisy quantum strategy, having the error value less than this value will establish the quantum advantage. This advantage of the qubit system over the cbit has been demonstrated experimentally [95].

3.8 Discussion

Our work brings to the forefront the efficacy of classical shared randomness in empowering the utility of classical communication. Although there has been a lot of research towards exploring the advantages of common-past resources to empower direct-communication resources, most of the investigations have been directed towards studying non-classical resources [7, 49]. At this point, it should be noted that nontrivial utility of classical shared randomness has also been studied in Reverse Shannon Theorem, both in classical and in quantum scenarios [96, 97, 48, 98]. In particular, the Ref.[48] is worth mentioning, where zero-error communication is studied using one classical channel to simulate another exactly in the presence of various classes of non-signalling correlations between sender and receiver *i.e.*, shared randomness, shared entanglement and arbitrary non-signalling correlations. It has been shown that while presence of classical shared randomness can be advantageous over the no correlation scenario, entanglement resources can provide further improvement over shared randomness. This in turn puts hierarchy among different kinds of additional resources in the form of shared correlations. Important to note that no limitation on the amount of shared correlation is considered in this analysis. Our results are one step forward in this respect as it consider only finite amount of shared randomness which leads to comparison among different amount of same type of shared correlation. For instance, perfect winning of our strict game is not possible with 1 cbit+ 1 bit SR, while $log3$ bit SR along with 1 cbit suffice the purpose.

Our proposed games are also important to reveal quantum advantage in the simplest communication scenario. While perfect winning of some games require shared randomness along with 1-bit of classical communication, in quantum scenario they can be won with 1-qubit communication alone. Importantly the advantage persists under experimental noises and hence welcomes novel experiments to implement the quantum protocol with presently available quantum technologies. The authors in Ref.[95] have experimentally demonstrated this quantum advantage. Our restaurant games along with their variant turn out to be useful for putting nontrivial 'order of merit' among different combinations of communication resources as tabled in Section 3.6.

The other important aspects are the no-go results established in Section 3.4.1, which prove that an advantage of qubit over classical bit must invoke quantum superposition both at the encoding and decoding steps. Our proposed game also opens up the opportunity to certify properties of different communication resources. Alice and Bob can certify the presence of direct communication resources by considering the $\mathbb{H}^n(1/n)$ game, as the input of Alice is completely random to Bob without any direct communication. If the direct communication resource is restricted to 1 cbit they will not be able to win $\mathbb{H}^n(1/n)$ game for any odd n without any preshared correlation. This can be useful in a practical scenario to verify the existence of preshared correlation between Alice and Bob by allowing them only 1 cbit direct communication.

Chapter 4

Study of resources in prepare and measure scenario

4.1 Introduction

Quantum information theory aims to harness the unique properties of quantum systems to develop communication protocols that are not achievable using classical Shannon theory [1]. For instance, in the quantum protocol of superdense coding, a sender (Alice) can transmit two classical bits of information to a distant receiver (Bob) using just a two-level quantum system, provided that Alice's system is entangled with a system held by Bob [7]. The presence of quantum entanglement enables this communication task, which is impossible in classical systems. However, without entanglement, the communication capabilities of quantum systems are fundamentally limited. For example, the Holevo theorem puts a constraint on the amount of classical information that can be extracted from a quantum system, limiting it to no more than n classical bits for n quantum bits [23]. A more recent and stringent result by Frenkel and Weiner demonstrates that the classical information storage capacity of a d-level quantum system is equivalent to that of a classical system with d states [24].

The aforementioned results indicate that a quantum system, without entanglement, may not be superior to its classical counterpart for transmitting classical information. However, the situation is more complex. For example, consider a communication task in the W-ANTV Scenario (see Section 2.3.2), where Alice and Bob receive random variables $x \in \mathscr{X}$ and $y \in \mathscr{Y}$ respectively, and Bob aims to compute a function $f(x, y)$. These scenarios, aimed at computing functions or relations f on $\mathscr{X} \times \mathscr{Y}$, are known as communication complexity. Ambainis *et al.* have considered the task in this scenario where a n-bit message is sent to Alice and Bob aims to recover an arbitrary bit with high success probability [26, 51]. They have shown that encoding the original message in the state of a 2-level quantum system (a qubit) yields higher success probability than the corresponding optimal strategies with a classical bit. When using classical resources, this task is commonly known as random access code (RAC), while the quantum version is called quantum-RAC (QRAC). Historically, the problem was first studied by Stephen Wiesner in a paper called "Conjugate Coding" [25]. However, the works of Ambainis *et al.* have renewed interest in this problem, and subsequent researchers have studied RAC/QRAC in connection with quantum communication complexity (see [29] and references therein), particularly in the context of network coding and locally decodable codes [99–104]. More recently, the problem of RAC and its variant, Parity-Oblivious Multiplexing, have been shown to have deep foundational implications [60, 105, 54, 106].

Although the benefits of quantum systems in reducing communication complexity have been extensively studied, not enough attention has been given to exploring the advantages of other operational theories. In this work, we consider a scenario where individual systems are assumed to be quantum, but their composition follows the NS condition. Even for two quantum systems, several consistent compositions are possible, with quantum being just one example. The state space of the resulting system lies between two extremes - the maximal tensor product state space and the minimal tensor product state space [107]. The maximal state space allows exotic joint states that are not allowed in quantum theory, while the minimal state space only allows separable states. It has been found that the system obtained through the maximal tensor product of two elementary quantum systems has advantages over quantum composition. This is noteworthy because all the NS correlations obtained from beyond quantum states are actually quantum simulable and therefore cannot yield nonlocal correlations beyond quantum [29]. Furthermore, we demonstrate that the minimal tensor product composition of two elementary quantum systems also provides an advantage in communication complexity. This is even more remarkable as the resulting theory is local by construction. Our work thus establishes a novel yet unexplored communication complexity advantage of different compositions of local quantum systems.

4.2 Random Access Code (RAC)

Random Access Code (RAC) is a distributed computational task where Alice receives a string of N independent bits, $\mathbf{a} = (a_1, \cdots, a_N)$ randomly sampled from

$\{0,1\}^N$, and Bob receives a random value of $b \in \{1,\cdots,N\}$. The data set **a** is encoded by sender, Alice, in smaller system of dimension d. Then the encoded system is sent to the receiver, Bob aims to retrieve the value of the b^{th} bit of Alice and output $\beta \in \{0,1\}$ as the guess for the random variable a_b (see Fig.4.1). They can also share classical shared randomness which can be utilized to correlate their strategies. we will call this (N,d)RAC game, and the average success probability of this task is given by

$$P_{N,d} = \frac{1}{N2^N} \sum_{a\in\{0,1\}^{\times N}} \sum_{b=1}^{N} P(\beta = a_b|a,b).$$

Another quantifier for the efficiency of the collaborative strategy for (N,d)RAC is also given by

$$I_{N,d} = \sum_{k=1}^{N} I\left(a_k : \beta \mid b = k\right), \tag{4.1}$$

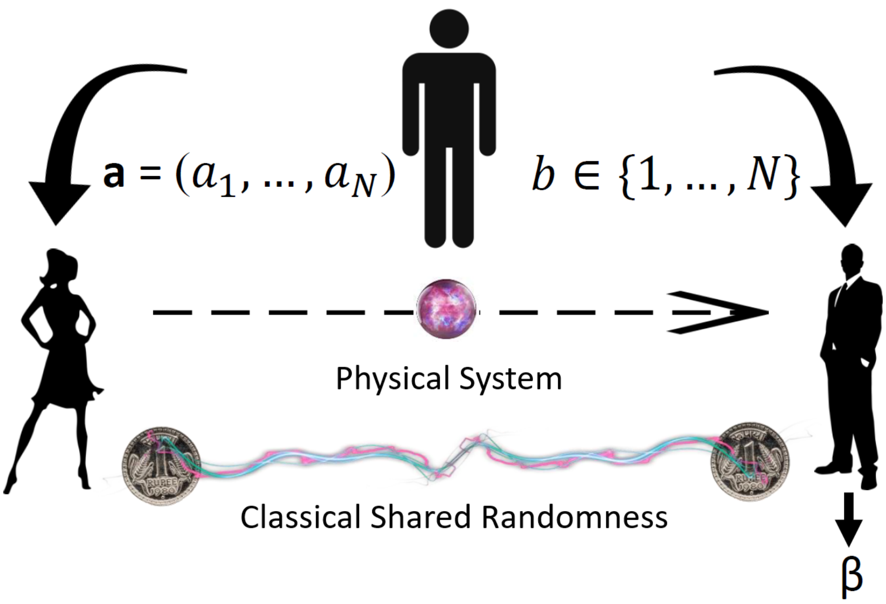


FIG. 4.1 [(N,d)RAC game] Referee provides a randomly chosen N-bit string $\mathbf{a} \in \{0,1\}^N$ to Alice and a random value $b \in \{1,\cdots,N\}$ to Bob. Bob aims to correctly guess the b^{th} bit of Alice's string, *i.e.*, to yield the outcome $\beta = a_b$. Alice and Bob can share classical Shared Randomness. Additionally, she can send some physical system carrying a bounded amount of information to Bob.

where $I(a_k : \beta \mid b = k)$ is the Shannon's mutual information between the k^{th} bit of Alice and Bob's guess β, computed under the condition that Bob has received $b = k$. If Alice communicates m-cbits to Bob then the efficiency is upper bounded by $I_{N,d} \leq m$. This condition is satisfied even when Alice communicates m-qubits to Bob with whom she may pre-share some classical correlation only, which we assume throughout the work.

4.3 Composition of elementary Quantum system

In the aforesaid game, Alice can communicate some abstract physical system to Bob. Within the mathematical framework of generalized probability theory (GPT) [28, 108–110] such an elementary system S can be specified by the tuple of normalized state and effect spaces, *i.e.* $\mathrm{S} \equiv (\Omega, \mathscr{E})$. Sometimes it is convenient to deal with unnormalized states and effects that form convex cones embedded in some $\mathbb{R}^n$. A GPT also captures the description of the composite system $\mathrm{S}^{AB} \equiv (\Omega^{AB}, \mathscr{E}^{AB})$ consisting of component subsystems $\mathrm{S}^A \equiv (\Omega^A, \mathscr{E}^A)$ and $\mathrm{S}^B \equiv (\Omega^B, \mathscr{E}^B)$. Under the restriction of NS and local tomography [111] the composite state space Ω^{AB} lies in between two extremes – (i) the maximal tensor product state space and (ii) the minimal tensor product state space [107]. For instance, the state cone of a quantum system associated with a Hilbert space $\mathscr{H}$ is the set of positive semidefinite operators $\mathscr{P}(\mathscr{H}) \subset \mathscr{L}(\mathscr{H})$ acting on $\mathscr{H}$, whereas the normalized states are the set of density operators $\mathscr{D}(\mathscr{H})$; here $\mathscr{L}(\mathscr{H})$ denotes the set of all linear operators acting on $\mathscr{H}$. For two quantum systems associated with Hilbert spaces $\mathscr{H}^A$ and $\mathscr{H}^B$ respectively, the state cone for maximal tensor product system is given by

$$\Omega_+^{AB}[\max] := \{\mathscr{W} \in \mathscr{L}(\mathscr{H}^A \otimes \mathscr{H}^B) | \operatorname{Tr}[\mathscr{W}(\pi_A \otimes \pi_B)] \geq 0 \\ \forall \pi_A \in \mathscr{P}(\mathscr{H}^A) \text{ and } \forall \pi_B \in \mathscr{P}(\mathscr{H}^B)\}.$$

Clearly, $\Omega_+^{AB}[\max]$ contains all the quantum states $\mathscr{P}(\mathscr{H}^A \otimes \mathscr{H}^B)$, and furthermore it encompasses states that are not allowed in quantum theory. For example, entanglement witnesses that are not bona-fide quantum states [41] are valid states in this composite model. The effect cone is constructed following the no-restriction hypothesis that allows all mathematically consistent effects in the theory [112]:

$$\mathscr{E}_+^{AB}[\max] := \{\pi \mid \pi = \sum_i \pi_i^A \otimes \pi_i^B;\ \pi_i^A \in \mathscr{P}(\mathscr{H}^A) \ \&\ \pi_i^B \in \mathscr{P}(\mathscr{H}^B)\}.$$

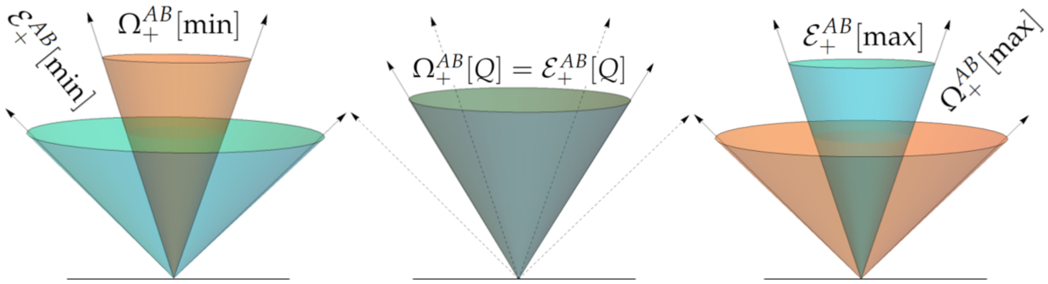


FIG. 4.2 Different possible compositions of two elementary quantum systems. Left: minimal tensor product composition – allows only separable states but effect cone is enlarged. Right: maximal tensor product composition – allows only separable effect but state cone is enlarged. Middle: quantum composition, state, and effect cones are identical (self-dual).

As it turns out $\mathscr{E}_+^{AB}[\max]$ also forms a cone which is dual to the state cone $\Omega_+^{AB}[\max]$. On the other extreme, the minimal tensor product contains only separable states, but the effect space gets enlarged here. More particularly, the role of state and effect cones of the maximal tensor product are interchanged in the minimal case, *i.e.*,

$$\Omega_+^{AB}[\min] := \mathscr{E}_+^{AB}[\max] \ \& \ \mathscr{E}_+^{AB}[\min] := \Omega_+^{AB}[\max].$$

Quantum composition lies in between and the state and effect cones becomes self dual in this case, *i.e.*, $\Omega_+^{AB}[Q] = \mathscr{P}(\mathscr{H}^A \otimes \mathscr{H}^B) = \mathscr{E}_+^{AB}[Q]$. The main objective of the present work is to see whether a particular composition can be more advantageous in (N,d)RAC game. At this point, it should be noted that all these three compositions have identical information capacity [113, 56, 114]. The information capacity of a system is defined as the maximal number of perfectly distinguishable states in a single shot measurement and intuitively it captures the amount of classical information that can be transferred perfectly by sending one copy of the system.

4.4 Advantage of extremal composition in measure and prepare scenario

Let us consider the $(3,d)$RAC game. If Alice communicates 2-cbits or 2-qubits (in quantum composition) to a classically correlated Bob then the efficiency of their collaborative strategy $I_{3,4}$ satisfies the bound $I_{3,4} \leq 2$. This bound follows from the satisfaction of the Information Causality principle [32]. Our next results show that this bound is not true in general if we consider other composition structures between two qubits.

Input string	Alice's encoding	Bob's decoding		
		$1^{st} \to$ XX	$2^{nd} \to$ YY	$3^{rd} \to$ ZZ
000	$\Gamma(\phi^+)$	1	1	1
001	ψ^+	1	1	0
010	ϕ^+	1	0	1
011	$\Gamma(\psi^+)$	1	0	0
100	ϕ^-	0	1	1
101	$\Gamma(\psi^-)$	0	1	0
110	$\Gamma(\phi^-)$	0	0	1
111	ψ^-	0	0	0
$I(a_k : \beta \mid b = k)$		1	1	1
$I_{3,4}$		$:= \sum_{k=1}^{3} I(a_k : \beta \mid b = k) = 3 > 2$		

TABLE 4.1 Prefect strategy for $(3,4)$RAC game with communication of two qubits in maximal tensor product. Here, $|\phi^\pm\rangle := (|00\rangle \pm |11\rangle)/\sqrt{2}$ and $|\psi^\pm\rangle := (|01\rangle \pm |10\rangle)/\sqrt{2}$. For each bit of Alice, Bob has a perfect decoding strategy yielding $I(a_k : \beta \mid b = k) = 1 \ \forall \ k \in \{1,2,3\}$. As a result, we have $I_{3,4} = 3 > 2$ and hence the resulting theory is more advantageous over quantum composition.

Theorem 6. *Maximal tensor product of two elementary quantum systems is more advantageous over their quantum composition in* $(3,4)$*RAC game.*

Proof. Let us denote $\Gamma(\psi_{AB}) := \mathbb{I}_A \otimes \mathrm{T}_B(\psi_{AB})$, where $\mathbb{I}$ is the identity map, T is the transpose map, and $\psi := |\psi\rangle\langle\psi|$. Clearly, for any $|\psi\rangle_{AB} \in \mathbb{C}^2 \otimes \mathbb{C}^2$ we have $\Gamma(\psi_{AB}) \in \Omega[\mathbb{C}^2 \otimes_{\max} \mathbb{C}^2]$ and hence in maximal tensor product theory such states are allowed for Alice's encodings. The explicit encodings are shown in Table 4.1. For decoding, Bob performs suitable product measurement on the encoded state received from Alice based on the bit $b \in \{1,2,3\}$ he obtained, and guesses the corresponding bit value based on the measurement outcome. For instance, by ZZ we denote the product measurement $\{|00\rangle_{AB}\langle 00|, |01\rangle_{AB}\langle 01|, |10\rangle_{AB}\langle 10|, |11\rangle_{AB}\langle 11|\}$ and we denote the coarse-grained outcome as 'c' (correlated) when projector on $|00\rangle$ or $|11\rangle$ clicks, else the outcome is denoted as 'ac' (anti-correlated). In Table 4.1 we also list the probability of obtaining correlated outcomes for three such decoding measurements (defined analogously) of Bob. It is now straightforward to see that the resulting encoding-decoding strategy more advantageous if Bob guesses the bit value 0 for the outcome 'c', else he guesses the value 1. □

The work of Ref.[115] is also worth mentioning at this point. There it has been shown that in the standard Bell experiment scenario all the NS correlations obtained in maximal tensor product theory are in fact quantum simulable hence beyond quantum nonlocal correlation is not possible there. Despite this, our Theorem 6 shows that maximal composition is more advantageous. In this respect our next result is even more striking as it deals with a model that contains only local correlations in Bell scenario.

Theorem 7. *Minimal tensor product of two elementary quantum systems is more advantageous over their quantum composition in* $(3,4)$*RAC game.*

Proof. In this case, only the product states of $\mathbb{C}^2 \otimes \mathbb{C}^2$ system are available to Alice for encoding her strings. The encoding scheme is listed in Table 4.2. However, Bob's decodings can be more versatile here as $\mathscr{E}^{AB}[\min] > \mathscr{E}^{AB}[Q] > \mathscr{E}^{AB}[\max]$. For

Input string	Alice's encoding	Bob's decoding		
		$1^{st} \to \mathscr{M}_1$	$2^{nd} \to \mathscr{M}_2$	$3^{rd} \to \mathscr{M}_3$
000	$\lvert+\rangle\lvert+\rangle$	1	1	1/2
001	$\lvert-\rangle\lvert-\rangle$	1	1	1/2
010	$\lvert 0\rangle\lvert 0\rangle$	1	0	1
011	$\lvert 1\rangle\lvert 1\rangle$	1	0	0
100	$\lvert 0\rangle\lvert 1\rangle$	0	1	0
101	$\lvert 1\rangle\lvert 0\rangle$	0	1	1
110	$\lvert+\rangle\lvert-\rangle$	0	0	1/2
111	$\lvert-\rangle\lvert+\rangle$	0	0	1/2
$I(a_k : \beta \mid b = k)$		1	1	≈ 0.19
$I_{3,4}$		$:= \sum_{k=1}^{3} I(a_k : \beta \mid b = k) \approx 2.19 > 2$		

TABLE 4.2 Advatage in $(3,4)$RAC game with minimal tensor product composition of two elementary qubits. Since the theory allows only product states the input strings are encoded in product states only. Here $\lvert\pm\rangle := (\lvert 0\rangle \pm \lvert 1\rangle)/\sqrt{2}$. In the rightmost three columns outcome probabilities of the effect E_i's are listed on the encoding states.

decoding the i^{th} bit Bob performs the measurement $\mathscr{M}_i := \{E_i, \bar{E}_i := \mathbb{I}_4 - E_i\}$, where

$$E_1 := \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \end{pmatrix}, \; E_2 := \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{pmatrix},$$

$$E_3 := \mathbb{I}_2 \otimes \frac{1}{2}(\mathbb{I}_2 + \sigma_z).$$

and guesses the bit value as 0 if the effect E_i clicks, else he guesses 1. Being a valid quantum measurement $\mathscr{M}_3$ is allowed measurement in minimal tensor product theory. Furthermore, on an arbitrary two-qubit product state $\rho_{mn} = \frac{1}{2}(\mathbb{I}_2 + \vec{m}.\sigma) \otimes \frac{1}{2}(\mathbb{I}_2 + \vec{n}.\sigma)$ we have, $\mathrm{Tr}[E_j \rho_{mn}] = \frac{1}{2}[1 + m_x n_x + (-1)^{(j+1)} m_z n_z] \geq 0$ and $\mathrm{Tr}[\bar{E}_j \rho_{mn}] = \frac{1}{2}[1 - m_x n_x - (-1)^{(j+1)} m_z n_z] \geq 0$ for $j \in \{1,2\}$, that in turn assure $\mathscr{M}_1$ & $\mathscr{M}_2$ to be the bona-fide measurements in minimal tensor product theory. The outcome probabilities of the effects E_i's on the encoded states are also listed in Table 4.2. The resulting encoding-decoding strategy yields $I_3 \approx 2.19 > 2$, establishing an advantage in $(3,4)$RAC game. □

Although the minimal composition of two elementary quantum systems is more advantageous (Theorem 7), but, unlike the maximal composition we could not come up with a perfect strategy for $(3,4)$RAC game here (see Theorem 6). In fact, at present, we do not know whether $I_{3,4} = 2.19$ is the optimal success of $(3,4)$RAC game with minimal composition of two qubits, and we leave this question here for future research.

4.5 Discusssion

The concept of composition is a key tool in shaping our worldview. While complex objects are typically made up of elementary components, certain compositions are considered implausible or counterintuitive [116]. This notion is also crucial when constructing theories in Physics [117–119]. In this work, we explore the role of different allowed compositions when dealing with multiple quantum systems. Notably, we demonstrate that, in the context of the $(3,4)$RAC task, different compositions of local quantum systems offer advantages over their quantum compositions. This advantage stems from the fact that communication tasks involve the preparation (for encoding) and measurement (for decoding) of the systems in question, making them more sensitive to structure. We have focused on the role of bipartite composition in communication tasks, leaving the study of multipartite compositions for future work.

Chapter 5

Certification of communication resources and their implications

5.1 Introduction

Correlations among distant events established through the violation of Bell type inequalities confirm nonlocal behavior of the physical world [14, 15, 62, 68]. Nonseparable multipartite quantum states yielding such correlations, in Schrödinger's words, are "...the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought" [12]. The advent of quantum information science identifies the power of such nonlocal correlations in numerous device-independent protocols – cryptographic key distribution [120], randomness certification [10] and amplification [11], dimension witness [121] are few canonical examples. To harness these resources in practical application, we must be able to certify their existence.

A fundamental approach to certifying quantum correlations, states and measurements in device-independent and semi-device-independent manner involves Bell inequalities. Let us consider a scenario where Alice and Bob share some bipartite system ρ_{AB} between them and the measurement choices of Alice and Bob are labeled by x and y yielding the output a and b respectively. If the correlation $P(a,b|x,y)$ obtained by them violates any Bell inequality *i.e.* $\mathscr{B}(P(a,b|x,y)) > \beta$ it will certify the entanglement of the shared state ρ_{AB} in a device-independent manner. The maximal violation of Bell CHSH inequality certifies the two-qubit maximally entangled states[122, 123]. Later a variety of quantum states have been proven to be self-testable e.g. All pure two-qubit states[124], maximally entangled pairs of qutrits[125], partially entangled pairs of qutrit violating CGLMP maximally[126, 127], etc. Even though the violation of Bell-CHSH inequality can certify the two-qubit maximally entangled states Cirel'son's result

[36], however, establishes that the nonlocal strength of quantum correlations is limited compared to the general *no-signalling* (NS) ones [37].

To better understand these fundamental limits of quantum nonlocality and explore possible extensions beyond standard quantum mechanics, researchers have proposed alternative theoretical frameworks. Here, we consider a class of theories wherein local measurements are described quantum mechanically, but they allow global structure more generic than quantum theory [128, 129, 115, 130–132]. Gleason-Busch celebrated result in quantum foundations proves that any map from generalized measurements to probability distributions can be written as the trace rule with the appropriate quantum state [133, 134] (see also [135]). This theorem, when appraised to the case of local observables acting on multipartite systems, hence called the unentangled Gleason's theorem, endorses the joint NS probability distributions to be obtained from some Hermitian operator called the positive over pure tensors (POPT) state [128, 129], which can also be obtained from Maximal tensor product of qubits. Although the set of POPT states is strictly larger than the set of quantum states (density operators), in a recent work, Barnum *et al.* have shown that the set of bipartite correlations attainable from the POPT states is precisely the set of quantum correlations [115]. This implies that certifying the maximal tensor products of local qubits (POPT) from quantum states is impossible in standard Bell scenario.

In this chapter, we analyze the correlations of multipartite POPT states obtained from local measurements performed on their constituent parts by considering a generalized Bell scenario as introduced in [66]. While in the standard Bell scenario, spatially separated parties receive some classical inputs and accordingly generate some classical outputs by performing local measurements on their respective parts of some composite system, recently Buscemi has generalized the scenario where the parties receive quantum inputs instead of classical variables [66]. In this generalized scenario, he has shown that all entangled states exhibit nonlocality, despite some of them allowing local-hidden-variable (LHV) model in classical input scenario [21, 136, 137]. Considering this generalized scenario, here we show that not all correlations obtained from bipartite POPT states are quantum simulable. In fact, every beyond quantum POPT state produces some beyond quantum correlation in some quantum input games. Our results paves the way for certifying the maximal tensor products of local qubits using this semiquantum games. On the other hand, to illustrate the limitations of the standard Bell scenario, we show that there are POPT states that produce classical-input-classical-output correlations that are not only quantum simulable but rather simulable classically. Our result shows that the *strong* claim made by the authors in [115] will not be correct anymore in this generalized Bell scenario which, as we will show, is allowed within the framework of local quantum theory.

These semi-quantum games provide a method for certifying states arising from the maximal composition of local quantum systems. In the previous chapter, we examined the communication advantages of both minimal and maximal compositions of local quantum systems within the framework of the RAC task. In this chapter we have also explores the foundational implications of these advantages. Specifically, we have demonstrated that both minimal and maximal compositions violate the principle of information causality [32].

5.2 Gleason's Theorem

We investigate the class of locally quantum theories studied in a series of works in the recent past [128, 129, 115, 130, 131]. In accordance with these works, we say that Alice is *locally quantum* if her physical system is described by a Hilbert space $\mathscr{H}_A$ with dimension d_A and her measurements M_A are given by a collection of effects corresponding to positive-operator-valued measurement (POVM) [138] operators $\{\pi_A^a\}_a$ acting on $\mathscr{H}_A$ and satisfying the constraint $\sum_a \pi_A^a = \mathbb{I}_A$; where $\forall\, a,\ \pi_A^a \in \mathscr{P}(\mathscr{H}_A) \subset \mathscr{L}(\mathscr{H}_A)$, with $\mathscr{P}(\mathscr{H}_A)$ and $\mathscr{L}(\mathscr{H}_A)$ respectively denoting the set of all positive operators and bounded linear operators acting on $\mathscr{H}_A$; and $\mathbb{I}_A$ is the identity operator on $\mathscr{H}_A$. The probability $p(a|M_A)$ that Alice obtains an outcome a for measurement $M_A \equiv \{\pi_A^a\}_a$ is given by a generalized probability measure $\mu : \mathscr{P}(\mathscr{H}_A) \mapsto [0,1]$, satisfying the properties (i) $\forall\ \pi_A^a \in \mathscr{P}(\mathscr{H}_A),\ 0 \leq \mu(\pi_A^a) \leq 1,$ (ii) $\mu(\mathbb{I}_A) = 1$, and (iii) $\mu(\sum_i \pi_A^i) = \sum_i \mu(\pi_A^i)$ for any sequence $\pi_A^1, \pi_A^2 \cdots$ with $\sum_i \pi_A^i \leq \mathbb{I}_A$. Each probability measure μ corresponds to a 'state' in the local quantum theory. We can make the association with the familiar quantum theory in which states are described by density operators by invoking the Gleason-Busch theorem according to which any such generalized probability measure is given by a linear functional of the form $\mu(\pi_A^a) = \mathrm{Tr}(\rho_A \pi_A^a)$, for some density operator $\rho_A \in \mathscr{D}(\mathscr{H}_A)$; $\mathscr{D}(\mathscr{H}_A)$ denotes the set of positive operators with unit-trace on $\mathscr{H}_A$.

Interesting situations arise when the theorem is generalized to the case of local observables acting on multipartite systems. Each party is assumed to be locally quantum as described above, with the i^{th} party performing the measurement $M_{A_i} \equiv \{\pi_{A_i}^a\}_a$. The 'state' is now given by a probability measure $\mu : \times_{i=1}^n \mathscr{P}(\mathscr{H}_{A_i}) \mapsto [0,1]$. According to Unentangled Gleason's theorem [128, 129], any such functional μ satisfying the no-signalling condition is of the form $\mu(\pi_{A_1}^{a_1},\cdots,\pi_{A_n}^{a_n}) = \mathrm{Tr}[W(\pi_{A_1}^{a_1}\otimes\cdots\otimes\pi_{A_n}^{a_n})]$, where W is a Hermitian, unit trace operator. Thus, the 'states' of multipartite locally quantum theory are in one-to-one correspondence with the operators W. W, being positive over all pure tensors, is called a POPT state. However, positivity of W over entangled effects is not assured and such a non-positive W can act as an entanglement witness operator [139]. We will denote the set of POPT states as $\mathscr{W}(\bigotimes_i \mathscr{H}_{A_i})$ which is same as the maximal tensor product of local quantum systems *i.e.* $\mathscr{W}(\bigotimes_i \mathscr{H}_{A_i}) \equiv \Omega_+[max]$ and it includes $\mathscr{D}(\bigotimes_i \mathscr{H}_{A_i})$ as a proper subset. W will be called 'beyond quantum state' (BQS) whenever $W \in \mathscr{W}(\bigotimes_i \mathscr{H}_{A_i}) \setminus \mathscr{D}(\bigotimes_i \mathscr{H}_{A_i})$. To certify the correlations obtained from BQSs, we briefly review the standard Bell scenario.

5.3 Standered Bell Scenario

A multipartite Bell scenario can be described as the following prover-verifier task. n distant verifiers $A_1, A_2, \cdots, A_n$ have their own source of classical indices $s_i \in S_i$. With the aim to verify some global property of a composite state prepared by a powerful but untrustworthy prover, they send their respective indices as inputs to spatially separated subsystems of the composite systems. Classical outputs $a_i \in O_i$ are generated from the different subsystems of the composite system and accordingly some payoff $\mathscr{P} : \times_{i=1}^{n}(S_i \times O_i) \mapsto \mathbb{R}$ is calculated. An implicit rule is that no communication is allowed among different subsystems once the game starts. Upon playing the game sufficiently many times, the input-output correlation $P \equiv \{p(a_1\cdots a_n|s_1\cdots s_n)\}_{s_i\in S_i}^{a_i\in O_i}$ is obtained. The collection of all NS correlations forms a convex polytope NS. A correlation is called classical if and only if it is of the form $p_L(a_1\cdots a_n|s_1\cdots s_n) = \int_\Lambda p(\lambda)\prod_i p(a_i|s_i,\lambda)d\lambda$, where $\lambda \in \Lambda$ is some classical variable shared among the parties. Collection of such correlations also forms a convex polytope L. On the other hand, a correlation is called quantum if it is obtained from some quantum state through local measurements, *i.e.* $p_Q(a_1\cdots a_n|s_1\cdots s_n) = \mathrm{Tr}[\rho(\bigotimes_i \pi_{s_i}^{a_i})]$ for some $\pi_{s_i}^{a_i} \in \mathscr{P}(\mathscr{H}_{A_i})$ and $\rho \in \mathscr{D}(\bigotimes_i \mathscr{H}_{A_i})$. The set of all quantum correlations Q forms a convex set but not a polytope. The framework of locally quantum theories allows us to define the correlation set obtained from the POPT states. Following the terminology of Ref. [130] we call such a correlation 'Gleason correlation' and denote the set as GL. The following set inclusion relations have been established: $L \subsetneq Q \subseteq GL \subsetneq NS$. While the first proper inclusion follows from the seminal work of Bell [14], the last one is due to Cirel'son and Popescu-Rohrlich [36, 37]. On the other hand the equality $Q = GL$ for bipartite correlations is established in [115]. More precisely, the authors in [115] have shown that for every POPT W_{AB} and for every local measurements $M_A = \{\pi_A^a\}_a$ and $M_B = \{\pi_B^b\}_b$, there exists a quantum state $\rho_{AB} \in \mathscr{D}(\mathscr{H}_A \otimes \mathscr{H}_B)$ and measurements $\tilde{M}_A = \{\tilde{\pi}_A^a\}_a$, $\tilde{M}_B = \{\tilde{\pi}_B^b\}_b$ such that, $\mathrm{Tr}[W_{AB}(\pi_A^a \otimes \pi_B^b)] = \mathrm{Tr}[\rho_{AB}(\tilde{\pi}_A^a \otimes \tilde{\pi}_B^b)]$. In this classical input-output scenario we are now in a position to prove our first result that in some sense can be considered stronger than the result of Barnum *et al* [115].

Proposition 3. *There exist beyond quantum bipartite states yielding correlations that are classically simulable.*

Proof. (Sketch) The family of operators $W_p := p\Gamma[|\phi^+\rangle\langle\phi^+|] + (1-p)\mathbb{I}/4$ is a BQS for $1/3 < p \leq 1$; $|\phi^+\rangle := (|00\rangle + |11\rangle)/\sqrt{2}$ and Γ denotes partial transposition. If we consider projective measurements only then a LHV description is possible whenever $p \leq 1/2$, whereas for generic POVMs one can have such a description for $p \leq 5/12$. The LHV models are motivated from the well known constructions of Werner [21] and Barrett [136]. We defer to the appendix A for the explicit construction. □

The result of Barnum *et al.* [115] and our Proposition 3 depicts the limitation of classical-input classical-output Bell scenario to reveal the full correlation strength of BQSs. This implies that in a standard Bell scenario, it is impossible to certify the BQS from input-output correlations. At this point, a more general Bell scenario turns out to be advantageous.

5.4 Semi Quantum Bell Scenario

This scenario was introduced by Buschemi to establish the nonlocal behavior of all entangled quantum states [66], which has subsequently generated a plethora of research interests [67, 140, 70, 141]. In this scenario, each of the verifiers, assumed to be *locally quantum,* has a random source of pure quantum states $\{\{p_i(s_i), \psi_{A_i^o}^{s_i}\} | s_i \in S_i\}$ (see Fig. 5.1). They wish to verify whether the state of a global system $W_{A_1 \cdots A_n}$, prepared by a powerful but untrustworthy prover, is BQS or not. To this aim, they provide their respective quantum states to the different parts of the distributed global state. The prover returns some classical index $a_i \in O_i$ by performing local quantum measurements $M_{A_iA_i^o} = \{\pi_{A_iA_i^o}^{a_i}\}_{a_i}$ on the respective distributed parts of the global state and the states received from the verifiers. Accordingly, some payoff $\beta : \times_{i=1}^n (S_i \times O_i) \mapsto \mathbb{R}$ is given, which specifies a semi-quantum game $\mathbb{G}_{sq}$. From the global state $W_{A_1 \cdots A_n}$, the prover can generate a correlation $P_{W_{A_1\cdots A_n}} := \{p(a_1,\cdots,a_n|\psi^{s_1},\cdots,\psi^{s_n})\}$ and the expected payoff is calculated as $\mathscr{I}_{\mathbb{G}_{sq}}(W_{A_1\cdots A_n}) := \sum_{s_1,a_1,\cdots,s_n,a_n} \beta(s_1,a_1,\cdots,s_n,a_n) \times p(a_1,\cdots,a_n|\psi^{s_1},\cdots,\psi^{s_n})$. Like the standard scenario, we can define the set of correlations $\mathscr{X}_{sq}$ with $\mathscr{X} \in \{L,Q,GL,NS\}$ and $\mathscr{X} \subseteq \mathscr{X}_{sq}$ in general. When the quantum sources consist of orthogonal quantum states, the scenario boils down to standard Bell scenario and no distinction is possible between a bipartite entangled state and a BQS [115].

In the Semiquantum Bell scenario, the prover performs local measurements $\{\pi^{a_i}_{A_iA_i^o}\}_{a_i}$ on the i^{th} subsystem. The composite multipartite state is given by a functional $\mu_{A_1A_1^o\cdots A_nA_n^o}$ which, invoking unentangled Gleason's theorem, corresponds to

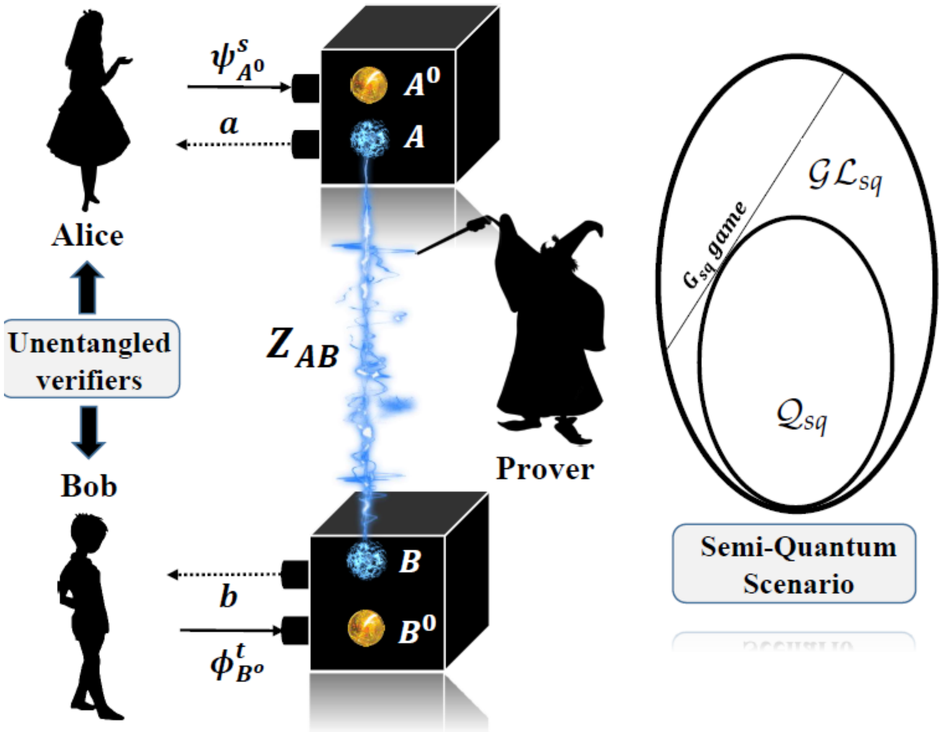


FIG. 5.1 A powerful but untrustworthy prover distributes a bipartite state Z_{AB} between two distant verifiers (Alice and Bob). The verifiers do not have any entanglement between them, but possess their own trusted local quantum preparation device. Such limited resourceful verifiers can verify the beyond quantumness of the state Z_{AB} provided to them (Theorem 8). The seminal Hahn-Banach separation theorem plays a crucial role in making this verification possible – the correlations produced from the bipartite quantum states form a convex-compact proper subset within the set of correlations produced from all bipartite states compatible with local quantum description and NS principle.

a POPT state $Z_{A_1A_1^o\cdots A_nA_n^o}$. It is in general not true that $Z_{A_1A_1^o\cdots A_nA_n^o}$ is given by the tensor product of the composite states. However, if the states held by the verifiers are pure and unentangled, then one can show that $Z_{A_1A_1^o\cdots A_nA_n^o} = \otimes_i \psi^{s_i}_{A_i^o} \otimes W_{A_1\cdots A_n}$. We leave the details in the appendix B. Interestingly, our next result shows that within the local quantum description, the unentangled verifiers (hence weakly resourceful) can test the property 'entangled vs BQS' supplied by the more resourceful prover.

Theorem 8. *For every beyond quantum state $W_{AB} \in \mathscr{W}(\mathscr{H}_A \otimes \mathscr{H}_B)$ there exists a semiquantum game $\mathbb{G}_{sq}$ such that $\mathscr{I}_{\mathbb{G}_{sq}}(W_{AB}) < 0$, while $\mathscr{I}_{\mathbb{G}_{sq}}(\rho_{AB}) \geq 0$, $\forall\ \rho_{AB} \in \mathscr{D}(\mathscr{H}_A \otimes \mathscr{H}_B)$.*

Proof. At the core of our proof lies the classic Hahn-Banach separation theorem of convex analysis and the fact that for every beyond quantum state $W_{AB} \in \mathscr{W}(\mathscr{H}_A \otimes \mathscr{H}_B)$ there exists an entangled state $\chi_{AB} \in \mathscr{D}(\mathscr{H}_A \otimes \mathscr{H}_B)$ such that $\mathrm{Tr}[W_{AB}\chi_{AB}] < 0$, whereas $\mathrm{Tr}[\sigma_{AB}\chi_{AB}] \geq 0$, $\forall\ \sigma_{AB} \in \mathscr{D}(\mathscr{H}_A \otimes \mathscr{H}_B)$ [142, 35, 143]. Also note that, there exits (non-unique) choices of pure states $\psi^s_A \in \mathscr{D}(\mathscr{H}_A)$ & $\psi^t_B \in \mathscr{D}(\mathscr{H}_B)$, and some real coefficients $\{\beta_{s,t}\}$ such that $\chi_{AB} = \sum_{s,t} \beta_{s,t} {\psi^s_A}^{\mathrm{T}} \otimes {\psi^t_B}^{\mathrm{T}}$ where T represents the transposition with respect to the computational basis. This leads us to the required game $\mathbb{G}^{\chi}_{sq}$ where the verifiers Alice and Bob yield quantum inputs $\psi^s_{A^o}$ and $\phi^t_{B^o}$, and ask the prover to return outputs $\in \{0,1\}$ from the distributed parts of the global state. The average payoff is calculated as $\mathscr{I} := \sum_{s,t} \beta_{s,t} p(11|\psi^s_{A^o}\psi^t_{B^o})$. The measurement $\{P^+_{uu^o}, \mathbb{I}_{uu^o} - P^+_{uu^o}\}$ is performed on the distributed parts of the global state, where $P^+_{uu^o} := |\phi^+\rangle_{uu^o}\langle\phi^+|$ with $|\phi^+\rangle_{uu^o} := \frac{1}{\sqrt{d_u}} \sum_{i=0}^{d_u-1} |ii\rangle$ and $P^+_{uu^o}$ corresponds to the outcome 1, $u \in \{A,B\}$. We therefore have,

$$\begin{aligned}\mathscr{I}_{\mathbb{G}^{\chi}_{sq}}(W_{AB}) &= \sum_{s,t} \beta_{s,t} \,\mathrm{Tr}\left[P^+_{AA^o} \otimes P^+_{BB^o} \left(\psi^s_{A^o} \otimes W_{AB} \otimes \psi^t_{B^o}\right)\right] \\ &= \sum_{s,t} \beta_{s,t} \,\mathrm{Tr}\left[(R_A \otimes R_B) W_{AB}\right];\end{aligned}$$

where R_A and R_B are the effective POVMs acting on the parts of Alice's and Bob's shares of the BQS, respectively; and are given by $R_u := \mathrm{Tr}_{u^o}[P^+_{uu^o}(\mathbb{I}_u \otimes \psi^s_{u^o})] = \frac{1}{d_u} {\psi^s_u}^{\mathrm{T}}$.

Therefore, we have

$$\begin{aligned}
\mathscr{I}_{\mathbb{G}^{\chi}_{sq}}(W_{AB}) &= \frac{1}{d_B d_A}\sum_{s,t}\beta_{s,t}\,\mathrm{Tr}\left[\left({\psi^s_A}^{\mathrm{T}}\otimes{\psi^t_B}^{\mathrm{T}}\right)W_{AB}\right]\\
&= \frac{1}{d_B d_A}\mathrm{Tr}\left[\left(\sum_{s,t}\beta_{s,t}\,{\psi^s_A}^{\mathrm{T}}\otimes{\psi^t_B}^{\mathrm{T}}\right)W_{AB}\right]\\
&= \frac{1}{d_B d_A}\mathrm{Tr}\left[\chi_{AB}W_{AB}\right] < 0.
\end{aligned}$$

On the other hand, given an arbitrary quantum state ρ_{AB} let the measurements $M_{AA^o} \equiv \{\pi^a_{AA^o}\}_a$ and $N_{BB^o} = \{\pi^b_{BB^o}\}_b$ be performed, where $a,b \in \{0,1\}$. The average payoff turns out to be

$$\begin{aligned}
\mathscr{I}_{\mathbb{G}^{\chi}_{sq}}(\rho_{AB}) &= \sum_{s,t}\beta_{s,t}\,\mathrm{Tr}\left[\pi^1_{AA^o}\otimes\pi^1_{BB^o}\left(\psi^s_{A^o}\otimes\rho_{AB}\otimes\psi^t_{B^o}\right)\right]\\
&= \sum_{s,t}\beta_{s,t}\,\mathrm{Tr}\left[R_{A^oB^o}\left(\psi^s_{A^o}\otimes\psi^t_{B^o}\right)\right],
\end{aligned}$$

where $R_{A^oB^o} := \mathrm{Tr}_{AB}[(\pi^1_{AA^o}\otimes\pi^1_{BB^o})(\mathbb{I}_{A^oB^o}\otimes\rho_{AB})]$ is a positive operator. Using linearity of trace we get,

$$\begin{aligned}
\mathscr{I}_{\mathbb{G}^{\chi}_{sq}}(\rho_{AB}) &= \mathrm{Tr}\left[R_{A^oB^o}\left(\sum_{s,t}\beta_{s,t}\,\psi^s_{A^o}\otimes\psi^t_{B^o}\right)\right]\\
&= \mathrm{Tr}\left[R_{A^oB^o}\ \chi^{\mathrm{T}}_{A^oB^o}\right] \geq 0.
\end{aligned}$$

The last inequality follows due to the fact that $\chi^{\mathrm{T}}_{A^oB^o}$ is a valid density operator, and this completes the proof. □

Theorem 8 establishes that $Q_{sq} \subsetneq GL_{sq}$ in the bipartite scenario. Note that, following an argument similar to [140], it can be shown that in the semiquantum scenario, even if classical communication between different distributed parts is allowed to the prover along with the quantum entangled state ρ_{AB}, still the local statistics obtained from BQS cannot be simulated. Our result poses some interesting questions. The proper set inclusion relation $Q \subsetneq NS$ established in [37] has motivated several novel approaches to isolate quantum correlations from beyond-quantum ones [144]. Along similar lines, the proper set inclusion relation $Q_{sq} \subsetneq GL_{sq}$ welcomes new principle(s) to isolate the quantum correlations from beyond-quantum ones in this generalized scenario. Importantly, our Theorem

8 suggests that such principles must be sensitive to the quantum signature of local inputs [71, 70].

The semi-quantum scenario also has important implications while studying correlations in multipartite (involving more than two parties) scenarios. Acín *et al.* have already pointed out that the result of Barnum *et al.* does not generalize to the multipartite scenario even in the classical-input classical-output paradigm [130]. They have provided examples of multipartite BQSs producing beyond quantum correlations within the standard Bell scenario. They have also pointed out that a BQS of the form

$$W_{A_1\cdots A_N} = \sum_k p_k (\Lambda^k_{A_1} \otimes \cdots \otimes \Lambda^k_{A_N})[\rho^k_{A_1\cdots A_n}], \tag{5.1}$$

will not generate any classical-input classical-output correlation that lies outside the set of correlations generated by quantum states. Here, $\{p_k\}$ is a probability distribution, $\rho^k_{A_1\cdots A_n} \in \mathscr{D}(\bigotimes_i \mathscr{H}_{A_i})$, and Λ^k_i are positive but not completely positive trace preserving maps on $\mathscr{L}(\mathscr{H}_{A_i})$ [143]. The authors in [130] have left the question open to identify the additional requirements to close the gap in their results. Our next result provides a solution to close this gap.

Theorem 9. *For every BQS $W_{A_1\cdots A_N} \in \mathscr{W}(\bigotimes_{i=1}^N \mathscr{H}_{A_i})$ there exists a semiquantum game $\mathbb{G}_{sq}$ such that $\mathscr{I}_{\mathbb{G}_{sq}}(W_{A_1\cdots A_N}) < 0$, whereas $\mathscr{I}_{\mathbb{G}_{sq}}(\rho_{A_1\cdots A_N}) \geq 0$, $\forall \rho_{A_1\cdots A_N} \in \mathscr{D}(\bigotimes_{i=1}^N \mathscr{H}_{A_i})$.*

The proof is a straightforward generalization of the proof of Theorem 8 (see appendix C). While Theorem 8 & 9 are just existence theorems, it is not hard to see that given an arbitrary BQS there is an efficient algorithm to construct a semiquantum game (the procedure is discussed in appendix D). It is important to note that non-orthogonal quantum inputs are necessary to reveal the beyond the quantum signature of correlation for any BQS of the form of Eq.(5.1). This implicitly follows from the results of Barnum *et al.* [115] and Acín *et el.* [130], the explicit prove is given is Appendix E. It is worth mentioning that this semi-quantum scenario is different from local tomography as it establishes beyond quantum nature of POPT states in a measurement device-independent manner where the measurement devices used to produce the classical outcomes need not be trusted [67].

5.5 Foundational Implication

Quantum mechanics is one of the most effective theories for describing nearly all natural phenomena, and its integration with information theory has led to numerous groundbreaking technologies, with the potential for even more advances [145]. Despite its successes, quantum mechanics has sparked extensive debates about its interpretation from its very inception [146–150], and these debates continue to this day [151, 152]. The formalism of quantum theory begins with an abstract mathematical framework in Hilbert space, raising the need for physical justification of its constructs. While the widely accepted no-signaling (NS) principle—prohibiting instantaneous communication between distant parties—plays an important role, it alone does not suffice to uniquely characterize quantum theory, as it allows a broad range of mathematical models to describe nature. Recently, inspired by developments in quantum information theory, several new principles have been proposed to address the limitations of the NS principle [19, 144, 32, 153–155, 94]. These principles efficiently rule out some beyond-quantum NS correlations as unphysical, thereby offering potential physical justifications for quantum correlations. However, correlations generated by maximal and minimal tensor products in Bell scenarios are quantum-simulable, indicating that these principles are insufficient for determining which composition accurately explains nature. While these correlations align in the Bell scenario, they differ in the prepare-and-measure scenario, motivating us to investigate these physical principles in prepare and measure scenario.

In this section, we analyze one of the intriguing principles called information causality (IC), proposed nearly a decade back [32]. IC can be envisaged as a generalization of the NS condition. It limits the information gain that a receiver (say Bob) can reach about a previously unknown to him data set of a sender (say Alice), by using two types of resources: (i) all his local resources that might be correlated with the sender *i.e.* Type-I resource (see section 2.2.1), and (ii) some physical system carrying bounded amount of information from Alice to Bob, *i.e.* Type-II resource (see section 2.2.2). Both these resources can further be of different kinds – classical, quantum, and beyond quantum; and IC principle provides a way to test their physicality. Although the correlated resources by themselves have no communication utility, as shown in the seminal superdense coding paper [7], a quantum correlation *viz.* entanglement can double up the communication capacity of a quantum channel. The power of entanglement, however, is limited in a way as it cannot enhance communication capacity of a classical channel. Principle of IC generalizes this no-go by limiting Bob's information gain to be at most m bits when m classical bits are communicated by Alice to him and he is allowed to use any of his local resources that might be correlated with Alice. Quite interestingly several NS correlations violate this principle and thus considered as unphysical [156–159]. In essence, restricting the Type-II resources to be classical, the IC principle discards some of the Type-I resources as unphysical.

Here we study the reverse scenario, *i.e.*, Type-II resource is varied among several possibilities while restricting the Type-I resource to be classical. In particular, we ask the question whether unphysicality of some Type-II resources can be established through information principles. Interestingly, as we will show, the principle of IC becomes effective in this case too. We consider the scenarios where individual systems are assumed to be quantum, but their composition can be anything in between maximal tensor product state space and minimal tensor product state space [107]. While the maximal one grants exotic joint states that are not allowed in quantum theory, the minimal one allows only separable states. It turns out that the system obtained through maximal tensor product of two elementary quantum violates the IC principle. This is quite remarkable as all the NS correlations obtained from beyond quantum states are quantum simulable and hence cannot yield beyond quantum nonlocal correlation [115]. We then show that minimal tensor product composition is also not compatible with IC principle and hence gets excluded. This is even more striking as the resulting theory allows only separable states and hence in Bell-type experiments no nonlocal correlation is possible. Our work establishes a novel yet unexplored proficiency of the IC principle as it discards extreme descriptions of composite quantum systems, and thus promises physical rationale for quantum composition.
Here we recall the concept of information capacity [89] (see also [113, 56, 114]), which is relevant for studying communication utility of a GPT system.

Definition 5 (CMP **316**, 441 (2012))**.** *Information capacity $\mathscr{I}(S)$ of a system S is the maximum number of states that can be perfectly distinguished in a single shot measurement.*

In standard information theoretic unit it is quantified in bit: if maximum d number of states are one-shot distinguishable then $\mathscr{I}(S) = \log d$ bit. Information capacity is closely related to the amount of classical information that can be transferred perfectly by sending one copy of the system. For instance, the state space of a two level classical and quantum systems are the 1-simplex and unit sphere in $\mathbb{R}^3$ respectively, and both have information capacity 2 (or $\log 2$ bit). However, it might happen that the Holevo capacity $\Theta(S)$ of a system S exceeds its information capacity. By Holevo capacity of a system we mean the optimal rate of transfer of classical information when asymptotically many copies of the system are in use. While finding Holevo capacity, the optimization is performed over all possible encodings and decodings allowed in the composite theory, *i.e.*, $\Theta(S) := \lim_{k\to\infty} \frac{1}{k} \log\left[\mathscr{I}(S^{\otimes k})\right]$. It turns out to be same as information capacity whenever information capacity is additive. While for quantum composition it is known to be additive, our next proposition first extends this additivity for minimal tensor product composition.

Proposition 4. *The information capacity of the minimal composition of k local quantum systems, each of dimension d, is d^k, i.e., $\mathscr{I}\left((\mathbb{C}^d)^{\otimes_{\min} k}\right) = d^k$.*

Proof. Assume that $\mathscr{I}\left((\mathbb{C}^d)^{\otimes_{\min} k}\right) = N$. Clearly, $N \geq d^k$ since there are d^k product states in $(\mathbb{C}^d)^{\otimes_{\min} k}$ that can be perfectly distinguished by product measurements. Furthermore, since we have assumed $\mathscr{I}\left((\mathbb{C}^d)^{\otimes_{\min} k}\right) = N$, there exist N pure product states $\{|a_1\rangle, \cdots, |a_N\rangle\}$ and a measurement $\{E_1, \cdots, E_N \mid \sum_{i=1}^N E_i = \mathbb{I}_d^{\otimes k}\}$ such that $\mathrm{Tr}(E_i |a_j\rangle\langle a_j|) = \delta_{ij}\ \ \forall i,j$. Let P_i be the projector on the orthogonal support of $|a_i\rangle\langle a_i|$, *i.e.*, $P_i = \mathbb{I}_d^{\otimes k} - |a_i\rangle\langle a_i|$. It is easy to show that P_i is a separable operator. To see this, first introduce an orthonormal basis for $\mathscr{H}_d^{\otimes k}$ consisting of pure product states such that $|a_i\rangle\langle a_i|$ is one of the elements of the basis set. Then, P_i can be written as the sum of projectors on all the basis vectors except $|a_i\rangle\langle a_i|$. We have,

$$
\begin{aligned}
d^k = \mathrm{Tr}(\mathbb{I}_d^{\otimes k}) &= \sum_{i=1}^N \mathrm{Tr}(E_i) = \sum_{i=1}^N \mathrm{Tr}[E_i \mathbb{I}_d^{\otimes k}] \\
&= \sum_{i=1}^N \mathrm{Tr}[E_i(|a_i\rangle\langle a_i| + P_i)] \\
&= \sum_{i=1}^N \mathrm{Tr}[E_i |a_i\rangle\langle a_i|] + \sum_{i=1}^N \mathrm{Tr}[E_i P_i] \\
&\geq \sum_{i=1}^N \mathrm{Tr}[E_i |a_i\rangle\langle a_i|] = \sum_{i=1}^N \delta_{ii} = N.
\end{aligned}
$$

The inequality follows from the fact that P_i is a separable operator, and thus $\mathrm{Tr}[E_i P_i] \geq 0$. We, therefore, conclude that $N = \mathscr{I}\left((\mathbb{C}^d)^{\otimes_{\min} k}\right) = d^k$. □

As an immediate implication it follows that the Holevo capacity of minimal tensor product of local quantum systems are same as their information capacity, *i.e.*, $\Theta(\mathbb{C}^d_{\min}) := \lim_{k\to\infty} \frac{1}{k} \log\left[\mathscr{I}((\mathbb{C}^d)^{\otimes_{\min} k})\right] = \lim_{k\to\infty} \frac{1}{k} \log(d^k) = \log d$. A similar result, as stated in the next proposition, also holds for maximal composition of locally quantum systems.

Proposition 5. *The information capacity of the maximal composition of k local quantum systems, each of dimension d, is d^k, i.e.,* $\mathscr{I}\left((\mathbb{C}^d)^{\otimes_{\max}k}\right) = d^k$.

Proof. Assume that $\mathscr{I}\left((\mathbb{C}^d)^{\otimes_{\max}k}\right) = N$. Clearly, $N \geq d^k$ since there are d^k product states in $(\mathbb{C}^d)^{\otimes_{\max}k}$ that can be perfectly distinguished by product measurements. Furthermore, since we have assumed $\mathscr{I}\left((\mathbb{C}^d)^{\otimes_{\max}k}\right) = N$, there exist N POPT states $\{W_1, \cdots, W_N\}$ and a separable measurement $\{E_1, \cdots, E_N \mid \sum_{i=1}^N E_i = \mathbb{I}_d^{\otimes k}\}$ such that $\mathrm{Tr}(E_i W_j) = \delta_{ij} \;\; \forall i, j$. Let $Y_i := \mathbb{I}_d^{\otimes k} - W_i$. We will first show that Y_i is an (unnormalized) POPT state, *i.e.*, $\langle a_{1\cdots k}| Y_i |a_{1\cdots k}\rangle \geq 0$ for any choice of product state $|a_{1\cdots k}\rangle := |a_1 \otimes a_2 \otimes \cdots \otimes a_k\rangle$. Indeed, $\langle a_{1\cdots k}| Y_i |a_{1\cdots k}\rangle = 1 - \langle a_{1\cdots k}| W_i |a_{1\cdots k}\rangle$, and we only need to show that $\langle a_{1\cdots k}| W_i |a_{1\cdots k}\rangle \leq 1$. But this follows from the fact that $\langle a_{1\cdots k}| W_i |a_{1\cdots k}\rangle$ must be a valid probability since W_i is a POPT state and $|a_1 \otimes a_2 \otimes \cdots \otimes a_k\rangle \langle a_1 \otimes a_2 \otimes \cdots \otimes a_k|$ is a valid effect in maximal composition. Now we can proceed in a similar manner as in the proof of Proposition 4, *i.e.*

$$
\begin{aligned}
d^k = \mathrm{Tr}(\mathbb{I}_d^{\otimes k}) &= \sum_{i=1}^N \mathrm{Tr}(E_i) = \sum_{i=1}^N \mathrm{Tr}[E_i \mathbb{I}_d^{\otimes k}] \\
&= \sum_{i=1}^N \mathrm{Tr}[E_i(W_i + Y_i)] \\
&= \sum_{i=1}^N \mathrm{Tr}[E_i W_i] + \sum_{i=1}^N \mathrm{Tr}[E_i Y_i] \\
&\geq \sum_{i=1}^N \mathrm{Tr}[E_i W_i] = \sum_{i=1}^N \delta_{ii} = N.
\end{aligned}
$$

The inequality follows from the fact that Y_i is POPT, and thus $\mathrm{Tr}[E_i Y_i] \geq 0$. Since we already showed that $N \geq d^k$, we conclude that $N = \mathscr{I}\left((\mathbb{C}^d)^{\otimes_{\max}k}\right) = d^k$. Consequently we have $\Theta(\mathbb{C}^d_{\max}) := \lim_{k\to\infty} \frac{1}{k} \log\left[\mathscr{I}((\mathbb{C}^d)^{\otimes_{\max}k})\right] = \lim_{k\to\infty} \frac{1}{k} \log(d^k) = \log d$. □

The objective of this work is to explore the foundational implication of communication advantages of several compositions of local quantum systems. In the following, we will show that several composition of local quantum systems violates IC principle.

Information Causality.– It is instructive to understand the IC principle through the distributed version of random access code task [51], which we recall from the last chapter (see Fig.4.1). Alice receives a string of N independent bits, $\mathbf{a} = (a_1, \cdots, a_N)$ randomly sampled from $\{0,1\}^N$, and Bob receives a random value of $b \in \{1, \cdots, N\}$. Bob's aim is to correctly guess the value of the b^{th} bit of Alice *i.e,* a_b. Alice can send some d dimensional system to assist Bob and as defined earlier this is the (N,d)RAC game, and the efficiency of the collaborative strategy is quantified through the quantity

$$I_{N,d} = \sum_{k=1}^{N} I\left(a_k : \beta \mid b = k\right), \tag{5.2}$$

where $I\left(a_k : \beta \mid b = k\right)$ is the Shannon's mutual information between the k^{th} bit of Alice and Bob's guess β, computed under the condition that Bob has received $b = k$. If Alice communicates d dimensional classical system to Bob then the efficiency is upper bounded by $I_N \leq \log d$. Furthermore, $I_N \leq \log d$ turns out to be a necessary condition for IC in an arbitrary theory where mutual information satisfies three abstract properties – (a) consistency, (b) data processing inequality, and (c) chain rule [32]. As pointed out by the authors in [32], IC holds true even if the quantum bits are transmitted provided that they are disentangled from the systems of the receiver – a consequence of Holevo bound, which limits information gain after transmission of m such qubits to m classical bits. While playing the $IC_{N,d}$ game by sending some GPT system from Alice to Bob in the presence of classical correlation shared in advance, the precise formulation of the IC, that we will use here, reads as $I_{N,d} \leq \Theta$, where Θ is the Holevo capacity of the communicated d dimensonal system in consideration.

Unphysicality of extremal quantum compositions.– Let us consider the $(3,4)$RAC game. If Alice communicates 2-cbits or 2-qubits (in quantum composition) to a classically correlated Bob then the necessary condition of IC is satisfied, *i.e.*, the efficiency of their collaborative strategy $I_{3,4}$ satisfies the bound $I_{3,4} \leq \log 4 = 2$. we will show that this is not true if one considers extremal tensor product compositions between two qubits.

Theorem 10. *Minimal tensor product composition of two elementary quantum violates IC principle.*

Proof. According to Proposition 4 the Holevo capacity of the minimal tensor product of two elementary quantum systems is 2 bits, so the necessary condition for satisfaction of IC principle is $I_{3,4} \leq 2$. But as discussed in the previous chapter there exists suitable encoding and decoding (see Table 4.2) leading to $I_{3,4} \approx 2.19 \geq 2$. □

At this point, it should be noted success of $(3,4)$RAC in Theorem 1 might not be the optimal one, and the question of optimality we leave here for the future. However, by considering the maximal tensor product composition between two qubits we know the existence of a perfect strategy for $(3,4)$RAC game.

Theorem 11. *Maximal tensor product composition of two elementary quantum violates IC principle.*

Proof. According to Proposition 5 the Holevo capacity of the maximal tensor product of two elementary quantum systems is 2 bits, so the necessary condition for satisfaction of IC principle is $I_{3,4} \leq 2$. But as discussed in the previous chapter there exists suitable encoding and decoding (see Table 4.1) leading to $I_{3,4} = 3 \geq 2$. This concludes the proof. □

While we have excluded the two extreme compositions through the violation of the IC principle, it would be interesting to exclude all other intermediate compositions except the quantum one in a similar approach. A first step in this direction is to generalize our Proposition 4 and 5 for such an arbitrary composition which we conjecture to hold true. Then an encoding-decoding strategy in such a composite model yielding a better than quantum success in (N,d)RAC game would invalidate that composition to be the physical one. For instance, any composition allowing states and effects used for encoding and decoding either in Theorem 10 or in Theorem 11 immediately gets excluded as unphysical.

5.6 Discussion

One of the most earnest research endeavors in quantum theory is to understand the limited nonlocal behavior of quantum correlations. Beyond its foundational significance, this question also holds practical relevance, as nonlocal correlations have been established as valuable resources for various tasks. In the previous chapter, we observed that several compositions of local quantum systems are beneficial for communication tasks. Our goal is to certify these compositions. However, in the bipartite scenario, Barnum et al. [115] demonstrated that all correlations are quantum-simulable. This result implies that the existence of a beyond-quantum state cannot be certified within the standard Bell scenario. To address this, we have considered the quantum-input Bell scenario. Our Theorem

8 establishes that for all bipartite beyond-quantum states, there exists a semi-quantum game capable of certifying the beyond-quantumness of the given state. We then extended our results to multipartite beyond-quantum states, enabling the certification of any POPT state. Additionally, in the previous chapter, we observed that the minimal and maximal compositions of local quantum systems provide greater advantages than quantum composition in the framework of the RAC task. We explored the deeper foundational implications of these advantages and demonstrated that both minimal and maximal compositions of local quantum systems violate the Information Causality (IC) principle.

Chapter 6

Summary and Further Directions

In this chapter, we summarize the key findings of this thesis and outline potential directions for future research. We have investigated the utility of various resources across different communication scenarios, examining both the certification of these resources and the foundational implications of their advantages in specific communication tasks. We investigate the simplest communication scenario involving only two parties. We consider general reward function(s) to quantify the communication utility of the physical system transferred from the sender to the receiver, with no classical shared randomness between the parties. First, we proposed a class of two-party games denoted as $\mathbb{H}^n(\gamma_1, \cdots, \gamma_n)$, where a game is specified by the parameters $\gamma_i \geq 0$ with $\sum_i \gamma_i = 1$. The special case where all the γ_i's are equal was designated with the simplified notation $\mathbb{H}^n(1/n)$. We have shown in Section 3.2.3 that only a measure zero subclass of these games can be won perfectly if 1-bit of classical communication is allowed from the sender (Alice) to the receiver (Bob). However, all such games can be won deterministically if the classical communication line is assisted by additional shared randomness. We have then shown in section 3.3 that a class of aforesaid games are perfectly winnable with 1-qubit communication from Alice to Bob that are otherwise not winnable with 1-bit of classical communication. This establishes a novel communication utility of the quantum system over its classical counterpart. To identify the origin of quantum advantage we have further proven two no-go results. We have shown the simultaneous use of quantum interference at the encoding step by Alice and at the decoding step by Bob is necessary for this particular advantage. We have analyzed the communication utility of a class of toy models, known as the polygon models [42]. Like two-level classical (bit) and quantum (qubit) systems, all these models allow at most two perfectly distinguishable states. However, with the help of our proposed game, we have shown in Section 3.4.2 that the communication utility of all the even-gons is strictly greater than the two-level classical system. We have then proposed a stricter version of the aforesaid game which we have denoted as $\mathbb{H}^n(1/(n-1))$. In section 3.5.1 we have proved that the $\mathbb{H}^4(1/3)$ game cannot be won perfectly with 1-bit of classical communication even when assisted with 1-bit of shared randomness. More shared randomness is required to achieve the goal. This provides further justification to consider shared randomness as a costly resource. Interestingly, we have then shown that $\mathbb{H}^4(1/3)$ game can be won with 1-qubit communication even without any assistance of classical shared randomness. Furthermore, no polygon system provides a perfect strategy for this game, making the advantage quantum sensitive (see section 3.5.2, section 3.5.3). In section 3.7 we have analyzed the robustness of the quantum protocols to noise so that this newly obtained communication advantage of the quantum system can be tested with imperfect experimental devices. Thereafter we have also considered the RAC task in the prepare and measure scenario and show that $(3,4)$RAC the quantum advantage is restricted but the maximal composition of local quantum systems is advantageous leading to perfect success probability. We have also shown that the minimal composition of local quantum system is also beneficial in this task over quantum composition, even though the correlations generated from this system are bell local. Furthermore, we have also studied the certification of these resources in a semi-device-independent manner. In section 5.4 we have demonstrated that for any bipartite beyond-quantum state (BQS), there exists a semi-quantum game capable of distinguishing the beyond-quantum state from quantum states. We have also extended this study for multipartite beyond quantum states (BQS) and shown the existence of a semi-quantum game for every BQS. We have also explored the foundational implication of this communication advantage in prepare and measure scenario by the minimal and maximal composition of local quantum systems. The information causality principle says that the total conditional information gained by the receiver about the sender's input should be bounded by Holevo capacity of the physical system. In section 5.5 we have derived the Holevo capacity of both minimal and maximal composition of local quantum systems. We have also shown that the minimal and maximal composition of the local quantum system violates the information causality principle. Here we also mention some relevant issues that could stem from the present thesis, as an interesting direction for further research.

1. We have shown in section 3.3 the existence of a task that can be won by using single qubit but 1cbit and $\log 2$ bit shared randomness is sufficient to win this task. Further studies can be done to come up with a game that can be won with 1-qubit communication. In contrast, in the classical case, an infinite amount of shared randomness, along with 1 -bit of classical communication, is required.

2.. We have also explored the source of this quantum advantage over classical bit, demonstrating that quantum superposition at the encoding and noncommutative measurement at decoding are necessary. At this point it might be interesting to recall an important result of quantum foundation recently established in [160]. While Bell in his seminal paper [14] and then Kochen & Specker [161] (see also Mermin [62, 162]) have shown that for the projective measurements on a two level quantum system, it is possible to come up with a 'deterministic hidden variable model', in [160] it has been shown that such a model is not possible when considering more general POVM measurement. It is worth interesting to study whether the quantum advantage observed here has some deeper connection with this no-go result. Additionally, several intriguing questions arise, such as whether shared randomness can enhance the effectiveness of other direct communication resources. Another open question is whether one can construct a communication scenario that establishes a trade-off between classical communication and shared randomness.

3. In section 4.4 we have shown that both the minimal and maximal composition of local quantum systems provide an advantage over their quantum composition in prepare and measure scenario. It will be interesting to find out whether all the other compositions are also advantageous in this scenario. We have explored the communication advantage of several compositions of two local quantum systems, it will be worthwhile to investigate the status of different compositions of multiple qubits.

4. In section 5.4 we have given a certification process for any states in maximal composition of two local quantum systems and also explored the certification process for composition of multiple qubits. A natural extension would be to examine the certification of states in the minimal composition of local quantum systems.

5. In section 5.5 we have explored the deeper foundational implication of communication advantage observed in different compositions. We have shown that both minimal and maximal composition of local quantum system violates information causality (IC) principle. A natural direction for future research is to investigate whether this communication advantage of different compositions could lead to a violation of the No-Hypersignaling principle [94].

References

[1] C. E. Shannon, "A mathematical theory of communication," *Bell System Technical Journal*, vol. 27, p. 379–423, july 1948.

[2] H. Nyquist, "Certain factors affecting telegraph speed1," *Bell System Technical Journal*, vol. 3, p. 324–346, Apr. 1924.

[3] R. V. L. Hartley, "Transmission of information1," *Bell System Technical Journal*, vol. 7, p. 535–563, July 1928.

[4] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik*. Springer Berlin Heidelberg, 1996.

[5] B. Schumacher, "Quantum coding," *Physical Review A*, vol. 51, p. 2738–2747, Apr. 1995.

[6] C. H. Bennett and G. Brassard, "Quantum cryptography: Public key distribution and coin tossing," *Proceedings of the IEEE International Conference on Computers, Systems and Signal Processing, Bangalore, India (IEEE, New York)*, p. 175–179, 1984.

[7] C. H. Bennett and S. J. Wiesner, "Communication via one- and two-particle operators on einstein-podolsky-rosen states," *Physical Review Letters*, vol. 69, p. 2881–2884, Nov. 1992.

[8] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, "Teleporting an unknown quantum state via dual classical and einstein-podolsky-rosen channels," *Phys. Rev. Lett.*, vol. 70, pp. 1895–1899, Mar 1993.

[9] M. Herrero-Collantes and J. C. Garcia-Escartin, "Quantum random number generators," *Reviews of Modern Physics*, vol. 89, Feb. 2017.

[10] S. Pironio, A. Acín, S. Massar, A. B. de la Giroday, D. N. Matsukevich, P. Maunz, S. Olmschenk, D. Hayes, L. Luo, T. A. Manning, and C. Monroe, "Random numbers certified by bell's theorem," *Nature*, vol. 464, p. 1021–1024, Apr. 2010.

[11] R. Colbeck and R. Renner, "Free randomness can be amplified," *Nature Physics*, vol. 8, p. 450–453, May 2012.

[12] E. Schrödinger, "Discussion of probability relations between separated systems," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 31, p. 555–563, Oct. 1935.

[13] A. Einstein, B. Podolsky, and N. Rosen, “Can quantum-mechanical description of physical reality be considered complete?,” *Physical Review*, vol. 47, p. 777–780, May 1935.

[14] J. S. Bell, “On the einstein podolsky rosen paradox,” *Physics Physique Fizika*, vol. 1, pp. 195–200, Nov 1964.

[15] J. S. BELL, “On the problem of hidden variables in quantum mechanics,” *Rev. Mod. Phys.*, vol. 38, pp. 447–452, Jul 1966.

[16] S. J. Freedman and J. F. Clauser, “Experimental test of local hidden-variable theories,” *Physical Review Letters*, vol. 28, p. 938–941, Apr. 1972.

[17] A. Aspect, P. Grangier, and G. Roger, “Experimental tests of realistic local theories via bell’s theorem,” *Physical Review Letters*, vol. 47, p. 460–463, Aug. 1981.

[18] M. Żukowski, A. Zeilinger, M. A. Horne, and A. K. Ekert, “““event-ready-detectors”’ bell experiment via entanglement swapping,” *Physical Review Letters*, vol. 71, p. 4287–4290, Dec. 1993.

[19] G. Brassard, H. Buhrman, N. Linden, A. A. Méthot, A. Tapp, and F. Unger, “Limit on nonlocality in any world in which communication complexity is not trivial,” *Physical Review Letters*, vol. 96, June 2006.

[20] C. Brukner, M. Żukowski, J.-W. Pan, and A. Zeilinger, “Bell’s inequalities and quantum communication complexity,” *Physical Review Letters*, vol. 92, Mar. 2004.

[21] R. F. Werner, “Quantum states with einstein-podolsky-rosen correlations admitting a hidden-variable model,” *Physical Review A*, vol. 40, p. 4277–4281, Oct. 1989.

[22] H. M. Wiseman, S. J. Jones, and A. C. Doherty, “Steering, entanglement, nonlocality, and the einstein-podolsky-rosen paradox,” *Physical Review Letters*, vol. 98, Apr. 2007.

[23] A. S. Holevo, “Bounds for the quantity of information transmitted by a quantum communication channel,” *Probl. Peredachi Inf.*, vol. 9, p. 177–183, 1973.

[24] P. Frenkel and M. Weiner, “Classical information storage in an n-level quantum system,” *The Bell System Technical Journal*, vol. 350, pp. 563–574, 2015.

[25] S. Wiesner, “Conjugate coding,” *ACM SIGACT News*, vol. 15, p. 78–88, Jan. 1983.

[26] A. Ambainis, A. Nayak, A. Ta-Shma, and U. Vazirani, “Dense quantum coding and a lower bound for 1-way quantum automata,” in *Proceedings of the thirty-first annual ACM symposium on Theory of Computing*, STOC99, ACM, May 1999.

[27] N. D. Mermin, "Extreme quantum entanglement in a superposition of macroscopically distinct states," *Physical Review Letters*, vol. 65, p. 1838–1840, Oct. 1990.

[28] J. Barrett, "Information processing in generalized probabilistic theories," *Physical Review A*, vol. 75, Mar. 2007.

[29] H. Buhrman, S. Cleve, R.and Massar, and R. de Wolf, "Nonlocality and communication complexity," *Rev. Mod. Phys.*, vol. 82, pp. 665–698, Mar 2010.

[30] A. Montanaro and R. de Wolf, "A survey of quantum property testing," *Theory of Computing*, vol. 1, no. 1, p. 1–81, 2016.

[31] I. Šupić and J. Bowles, "Self-testing of quantum systems: a review," *Quantum*, vol. 4, p. 337, Sept. 2020.

[32] M. Pawłowski, T. Paterek, D. Kaszlikowski, V. Scarani, A. Winter, and M. Żukowski, "Information causality as a physical principle," *Nature*, vol. 461, p. 1101–1104, Oct. 2009.

[33] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition*. Cambridge University Press, June 2012.

[34] A. Peres, "Separability criterion for density matrices," *Physical Review Letters*, vol. 77, p. 1413–1415, Aug. 1996.

[35] M. Horodecki, P. Horodecki, and R. Horodecki, "Separability of mixed states: necessary and sufficient conditions," *Physics Letters A*, vol. 223, p. 1–8, Nov. 1996.

[36] B. S. Cirel'son, "Quantum generalizations of bell's inequality," *Letters in Mathematical Physics*, vol. 4, p. 93–100, Mar. 1980.

[37] S. Popescu and D. Rohrlich, "Quantum nonlocality as an axiom," *Foundations of Physics*, vol. 24, p. 379–385, Mar. 1994.

[38] R. K. Patra, S. G. Naik, E. P. Lobo, S. Sen, T. Guha, S. S. Bhattacharya, M. Alimuddin, and M. Banik, "Classical analogue of quantum superdense coding and communication advantage of a single quantum system," *Quantum*, vol. 8, p. 1315, Apr. 2024.

[39] R. K. Patra, S. G. Naik, E. P. Lobo, S. Sen, G. L. Sidhardh, M. Alimuddin, and M. Banik, "Principle of information causality rationalizes quantum composition," *Physical Review Letters*, vol. 130, Mar. 2023.

[40] E. P. Lobo, S. G. Naik, S. Sen, R. Patra, M. Banik, and M. Alimuddin, "Certifying beyond quantumness of locally quantum no-signaling theories through a quantum-input bell test," *Phys. Rev. A*, vol. 106, p. L040201, Oct 2022.

[41] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, “Quantum entanglement,” *Reviews of Modern Physics*, vol. 81, p. 865–942, June 2009.

[42] P. Janotta, C. Gogolin, J. Barrett, and N. Brunner, “Limits on nonlocal correlations from the structure of the local state space,” *New Journal of Physics*, vol. 13, p. 063024, June 2011.

[43] M. J. W. Hall, “Relaxed bell inequalities and kochen-specker theorems,” *Phys. Rev. A*, vol. 84, p. 022102, Aug 2011.

[44] T. Schaetz, M. D. Barrett, D. Leibfried, J. Chiaverini, J. Britton, W. M. Itano, J. D. Jost, C. Langer, and D. J. Wineland, “Quantum dense coding with atomic qubits,” *Phys. Rev. Lett.*, vol. 93, p. 040505, Jul 2004.

[45] J. Barreiro, T. Wei, and P. Kwiat, “Beating the channel capacity limit for linear photonic superdense coding,” *Nature Phys*, vol. 4, p. 282–286, 2008.

[46] B. P. Williams, R. J. Sadlier, and T. S. Humble, “Superdense coding over optical fiber links with complete bell-state measurements,” *Phys. Rev. Lett.*, vol. 118, p. 050501, Feb 2017.

[47] C. H. Bennett, P. W. Shor, J. A. Smolin, and A. V. Thapliyal, “Entanglement-assisted classical capacity of noisy quantum channels,” *Phys. Rev. Lett.*, vol. 83, pp. 3081–3084, Oct 1999.

[48] T. S. Cubitt, D. Leung, W. Matthews, and A. Winter, “Zero-error channel capacity and simulation assisted by non-local correlations,” *IEEE Transactions on Information Theory*, vol. 57, no. 8, pp. 5509–5523, 2011.

[49] P. E. Frenkel and M. Weiner, “On entanglement assistance to a noiseless classical channel,” *Quantum*, vol. 6, p. 662, Mar. 2022.

[50] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, “Proposed experiment to test local hidden-variable theories,” *Physical Review Letters*, vol. 23, p. 880–884, Oct. 1969.

[51] A. Ambainis, A. Nayak, A. Ta-Shma, and U. Vazirani, “Dense quantum coding and quantum finite automata,” *Journal of the ACM*, vol. 49, p. 496–511, July 2002.

[52] M. Banik, S. S. Bhattacharya, A. Mukherjee, A. Roy, A. Ambainis, and A. Rai, “Limited preparation contextuality in quantum theory and its relation to the cirel’son bound,” *Physical Review A*, vol. 92, Sept. 2015.

[53] L. Czekaj, M. Horodecki, P. Horodecki, and R. Horodecki, “Information content of systems as a physical principle,” *Physical Review A*, vol. 95, Feb. 2017.

[54] A. Ambainis, M. Banik, A. Chaturvedi, D. Kravchenko, and A. Rai, “Parity oblivious d-level random access codes and class of noncontextuality inequalities,” *Quantum Information Processing*, vol. 18, Mar. 2019.

[55] V. M, R. k. Patra, M. Janpandit, S. Sen, M. Banik, and A. Chaturvedi, "Mutually unbiased balanced functions and generalized random access codes," *Phys. Rev. A*, vol. 104, p. 012420, Jul 2021.

[56] S. G. Naik, E. P. Lobo, S. Sen, R. K. Patra, M. Alimuddin, T. Guha, S. S. Bhattacharya, and M. Banik, "Composition of multipartite quantum systems: Perspective from timelike paradigm," *Phys. Rev. Lett.*, vol. 128, p. 140401, Apr 2022.

[57] A. Ambainis, D. Leung, L. Mancinska, and M. Ozols, "Quantum random access codes with shared randomness," 2009.

[58] M. Pawłowski and M. Żukowski, "Entanglement-assisted random access codes," *Phys. Rev. A*, vol. 81, p. 042326, Apr 2010.

[59] A. Tavakoli, J. Pauwels, E. Woodhead, and S. Pironio, "Correlations in entanglement-assisted prepare-and-measure scenarios," *PRX Quantum*, vol. 2, p. 040357, Dec 2021.

[60] R. W. Spekkens, D. H. Buzacott, A. J. Keehn, B. Toner, and G. J. Pryde, "Preparation contextuality powers parity-oblivious multiplexing," *Phys. Rev. Lett.*, vol. 102, p. 010401, Jan 2009.

[61] W. van Dam, "Nonlocality & communication complexity," *Ph.D. thesis, Department of Physics, University of Oxford*, 1999.

[62] N. D. Mermin, "Hidden variables and the two theorems of john bell," *Reviews of Modern Physics*, vol. 65, p. 803–815, July 1993.

[63] W. Slofstra, "Tsirelson's problem and an embedding theorem for groups arising from non-local games," *Journal of the American Mathematical Society*, vol. 33, p. 1–56, Sept. 2019.

[64] Z. Ji, A. Natarajan, T. Vidick, J. Wright, and H. Yuen, "Mip*=re," 2020.

[65] T. Fritz, "Quantum logic is undecidable," *Archive for Mathematical Logic*, vol. 60, p. 329–341, Sept. 2020.

[66] F. Buscemi, "All entangled quantum states are nonlocal," *Physical Review Letters*, vol. 108, May 2012.

[67] C. Branciard, D. Rosset, Y.-C. Liang, and N. Gisin, "Measurement-device-independent entanglement witnesses for all entangled quantum states," *Physical Review Letters*, vol. 110, Feb. 2013.

[68] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, "Bell nonlocality," *Rev. Mod. Phys.*, vol. 86, pp. 419–478, Apr 2014.

[69] E. Wolfe, D. Schmid, A. B. Sainz, R. Kunjwal, and R. W. Spekkens, "Quantifying bell: the resource theory of nonclassicality of common-cause boxes," *Quantum*, vol. 4, p. 280, June 2020.

[70] D. Schmid, D. Rosset, and F. Buscemi, “The type-independent resource theory of local operations and shared randomness,” *Quantum*, vol. 4, p. 262, Apr. 2020.

[71] D. Rosset, D. Schmid, and F. Buscemi, “Type-independent characterization of spacelike separated resources,” *Phys. Rev. Lett.*, vol. 125, p. 210402, Nov 2020.

[72] R. J. Aumann, “Correlated equilibrium as an expression of bayesian rationality,” *Econometrica*, vol. 55, p. 1, Jan. 1987.

[73] L. Babai and P. Kimmel, “Randomized simultaneous messages: solution of a problem of yao in communication complexity,” in *Proceedings of Computational Complexity. Twelfth Annual IEEE Conference*, CCC-97, IEEE Comput. Soc, 1997.

[74] C. L. Canonne, V. Guruswami, R. Meka, and M. Sudan, “Communication with imperfectly shared randomness,” *IEEE Transactions on Information Theory*, vol. 63, no. 10, pp. 6799–6818, 2017.

[75] B. F. Toner and D. Bacon, “Communication cost of simulating bell correlations,” *Phys. Rev. Lett.*, vol. 91, p. 187904, Oct 2003.

[76] J. Bowles, F. Hirsch, M. Quintino, and N. Brunner, “Local hidden variable models for entangled quantum states using finite shared randomness,” *Phys. Rev. Lett.*, vol. 114, p. 120401, Mar 2015.

[77] M. Perarnau-Llobet, K. V. Hovhannisyan, M. Huber, P. Skrzypczyk, N. Brunner, and A. Acín, “Extractable work from correlations,” *Phys. Rev. X*, vol. 5, p. 041011, Oct 2015.

[78] T. Guha, M. Alimuddin, S. Rout, A. Mukherjee, S. S. Bhattacharya, and M. Banik, “Quantum advantage for shared randomness generation,” *Quantum*, vol. 5, p. 569, Oct. 2021.

[79] J. F. Nash, “Equilibrium points in n -person games,” *Proceedings of the National Academy of Sciences*, vol. 36, p. 48–49, Jan. 1950.

[80] J. C. Harsanyi, “Games with incomplete information played by “bayesian” players, i–iii part i. the basic model,” *Management Science*, vol. 14, p. 159–182, Nov. 1967.

[81] J. C. Harsanyi, “Games with incomplete information played by “bayesian” players part ii. bayesian equilibrium points,” *Management Science*, vol. 14, p. 320–334, Jan. 1968.

[82] J. C. Harsanyi, “Games with incomplete information played by ‘bayesian’ players, part iii. the basic probability distribution of the game,” *Management Science*, vol. 14, p. 486–502, Mar. 1968.

[83] C. H. Papadimitriou and T. Roughgarden, “Computing correlated equilibria in multi-player games,” *Journal of the ACM*, vol. 55, p. 1–29, July 2008.

[84] N. Brunner and N. Linden, “Connection between bell nonlocality and bayesian game theory,” *Nature Communications*, vol. 4, July 2013.

[85] A. Pappa, N. Kumar, T. Lawson, M. Santha, S. Zhang, E. Diamanti, and I. Kerenidis, “Nonlocality and conflicting interest games,” *Phys. Rev. Lett.*, vol. 114, p. 020401, Jan 2015.

[86] A. Roy, A. Mukherjee, T. Guha, S. Ghosh, S. S. Bhattacharya, and M. Banik, “Nonlocal correlations: Fair and unfair strategies in bayesian games,” *Phys. Rev. A*, vol. 94, p. 032120, Sep 2016.

[87] M. Banik, S. S. Bhattacharya, N. Ganguly, T. Guha, A. Mukherjee, A. Rai, and A. Roy, “Two-qubit pure entanglement as optimal social welfare resource in bayesian game,” *Quantum*, vol. 3, p. 185, Sept. 2019.

[88] S. Massar and M. K. Patra, “Information and communication in polygon theories,” *Physical Review A*, vol. 89, May 2014.

[89] M. P. Müller and C. Ududec, “Structure of reversible computation determines the self-duality of quantum theory,” *Phys. Rev. Lett.*, vol. 108, p. 130401, Mar 2012.

[90] S. W. Al-Safi and J. Richens, “Reversibility and the structure of the local state space,” *New Journal of Physics*, vol. 17, p. 123001, Dec. 2015.

[91] M. Banik, S. Saha, T. Guha, S. Agrawal, S. S. Bhattacharya, A. Roy, and A. S. Majumdar, “Constraining the state space in any physical theory with the principle of information symmetry,” *Phys. Rev. A*, vol. 100, p. 060101, Dec 2019.

[92] S. Saha, S. S. Bhattacharya, T. Guha, S. Halder, and M. Banik, “Advantage of quantum theory over nonclassical models of communication,” *Annalen der Physik*, vol. 532, Nov. 2020.

[93] S. S. Bhattacharya, S. Saha, T. Guha, and M. Banik, “Nonlocality without entanglement: Quantum theory and beyond,” *Phys. Rev. Res.*, vol. 2, p. 012068, Mar 2020.

[94] M. Dall’Arno, S. Brandsen, A. Tosini, F. Buscemi, and V. Vedral, “No-hypersignaling principle,” *Phys. Rev. Lett.*, vol. 119, p. 020401, Jul 2017.

[95] C. Ding, E. P. Lobo, M. Alimuddin, X.-Y. Xu, S. Zhang, M. Banik, W.-S. Bao, and H.-L. Huang, “Quantum advantage: A single qubit’s experimental edge in classical data storage,” *Physical Review Letters*, vol. 133, Nov. 2024.

[96] A. Winter, “Compression of sources of probability distributions and density operators,” 2002.

[97] C. Bennett, P. Shor, J. Smolin, and A. Thapliyal, “Entanglement-assisted capacity of a quantum channel and the reverse shannon theorem,” *IEEE Transactions on Information Theory*, vol. 48, no. 10, pp. 2637–2655, 2002.

[98] C. H. Bennett, I. Devetak, A. W. Harrow, P. W. Shor, and A. Winter, "The quantum reverse shannon theorem and resource tradeoffs for simulating quantum channels," *IEEE Transactions on Information Theory*, vol. 60, no. 5, pp. 2926–2959, 2014.

[99] H. Klauck, "Lower bounds for quantum communication complexity," in *Proceedings 42nd IEEE Symposium on Foundations of Computer Science*, p. 288–297, IEEE, 2001.

[100] I. Kerenidis and R. de Wolf, "Exponential lower bound for 2-query locally decodable codes via a quantum argument," in *Proceedings of the thirty-fifth annual ACM symposium on Theory of computing*, STOC03, p. 106–115, ACM, June 2003.

[101] S. Aaronson, "Limitations of quantum advice and one-way communication," *Theory of Computing*, vol. 1, no. 1, p. 1–28, 2005.

[102] S. Wehner and R. de Wolf, *Improved Lower Bounds for Locally Decodable Codes and Private Information Retrieval*, p. 1424–1436. Springer Berlin Heidelberg, 2005.

[103] D. Gavinsky, J. Kempe, O. Regev, and R. de Wolf, "Bounded-error quantum state identification and exponential separations in communication complexity," *SIAM Journal on Computing*, vol. 39, p. 1–24, Jan. 2009.

[104] M. Hayashi, K. Iwama, H. Nishimura, R. Raymond, and S. Yamashita, *Quantum Network Coding*, p. 610–621. Springer Berlin Heidelberg, 2007.

[105] A. Chailloux, I. Kerenidis, S. Kundu, and J. Sikora, "Optimal bounds for parity-oblivious random access codes," *New Journal of Physics*, vol. 18, p. 045003, Apr. 2016.

[106] A. Chaturvedi and D. Saha, "Quantum prescriptions are more ontologically distinct than they are operationally distinguishable," *Quantum*, vol. 4, p. 345, Oct. 2020.

[107] I. Namioka and R. Phelps, "Tensor products of compact convex sets," *Pacific Journal of Mathematics*, vol. 31, p. 469–480, Nov. 1969.

[108] G. Chiribella, G. M. D'Ariano, and P. Perinotti, "Informational derivation of quantum theory," *Physical Review A*, vol. 84, July 2011.

[109] H. Barnum and A. Wilce, "Information processing in convex operational theories," *Electronic Notes in Theoretical Computer Science*, vol. 270, p. 3–15, Feb. 2011.

[110] L. Masanes and M. P. Müller, "A derivation of quantum theory from physical requirements," *New Journal of Physics*, vol. 13, p. 063001, June 2011.

[111] H. Barnum and A. Wilce, "Local tomography and the jordan structure of quantum theory," *Foundations of Physics*, vol. 44, p. 192–212, Feb. 2014.

[112] G. Chiribella, G. M. D'Ariano, and P. Perinotti, "Probabilistic theories with purification," *Physical Review A*, vol. 81, June 2010.

[113] H. Arai, Y. Yoshida, and M. Hayashi, "Perfect discrimination of non-orthogonal separable pure states on bipartite system in general probabilistic theory," *Journal of Physics A: Mathematical and Theoretical*, vol. 52, p. 465304, Oct. 2019.

[114] S. Sen, E. P. Lobo, R. K. Patra, S. G. Naik, A. Das Bhowmik, M. Alimuddin, and M. Banik, "Timelike correlations and quantum tensor product structure," *Physical Review A*, vol. 106, Dec. 2022.

[115] H. Barnum, S. Beigi, S. Boixo, M. B. Elliott, and S. Wehner, "Local quantum measurement and no-signaling imply quantum correlations," *Physical Review Letters*, vol. 104, Apr. 2010.

[116] L. S. PENROSE and R. PENROSE, "Impossible objects: A special type of visual illusion," *British Journal of Psychology*, vol. 49, p. 31–33, Feb. 1958.

[117] L. Hardy, *On the Theory of Composition in Physics*, p. 83–106. Springer Berlin Heidelberg, 2013.

[118] L. Hardy, "Operational general relativity: Possibilistic, probabilistic, and quantum," 2016.

[119] A. D. Bhowmik, P. Parashar, G. Kar, and M. Banik, "From no causal loop to absoluteness of cause: discarding the quantum not logic," 2021.

[120] J. Barrett, L. Hardy, and A. Kent, "No signaling and quantum key distribution," *Physical Review Letters*, vol. 95, June 2005.

[121] N. Brunner, S. Pironio, A. Acin, N. Gisin, A. A. Méthot, and V. Scarani, "Testing the dimension of hilbert spaces," *Physical Review Letters*, vol. 100, May 2008.

[122] S. J. Summers and R. Werner, "Maximal violation of bell's inequalities is generic in quantum field theory," *Communications in Mathematical Physics*, vol. 110, p. 247–259, June 1987.

[123] S. Popescu and D. Rohrlich, "Which states violate bell's inequality maximally?," *Physics Letters A*, vol. 169, p. 411–414, Oct. 1992.

[124] A. Coladangelo, K. T. Goh, and V. Scarani, "All pure bipartite entangled states can be self-tested," *Nature Communications*, vol. 8, May 2017.

[125] A. Salavrakos, R. Augusiak, J. Tura, P. Wittek, A. Acín, and S. Pironio, "Bell inequalities tailored to maximally entangled states," *Phys. Rev. Lett.*, vol. 119, p. 040402, Jul 2017.

[126] T. H. Yang, T. Vértesi, J. D. Bancal, V. Scarani, and M. Navascués, "Robust and versatile black-box certification of quantum devices," *Phys. Rev. Lett.*, vol. 113, p. 040401, Jul 2014.

[127] A. Acín, T. Durt, N. Gisin, and J. I. Latorre, "Quantum nonlocality in two three-level systems," *Phys. Rev. A*, vol. 65, p. 052325, May 2002.

[128] N. R. Wallach, "An unentangled gleason's theorem," 2000.

[129] H. Barnum, C. A. Fuchs, J. M. Renes, and A. Wilce, "Influence-free states on compound quantum systems," 2005.

[130] A. Acín, R. Augusiak, D. Cavalcanti, C. Hadley, J. K. Korbicz, M. Lewenstein, L. Masanes, and M. Piani, "Unified framework for correlations in terms of local quantum observables," *Physical Review Letters*, vol. 104, Apr. 2010.

[131] G. de la Torre, L. Masanes, A. J. Short, and M. P. Müller, "Deriving quantum theory from its local structure and reversibility," *Physical Review Letters*, vol. 109, Aug. 2012.

[132] M. Kleinmann, T. J. Osborne, V. B. Scholz, and A. H. Werner, "Typical local measurements in generalized probabilistic theories: Emergence of quantum bipartite correlations," *Physical Review Letters*, vol. 110, Jan. 2013.

[133] A. Gleason, "Measures on the closed subspaces of a hilbert space," *Indiana University Mathematics Journal*, vol. 6, no. 4, p. 885–893, 1957.

[134] P. Busch, "Quantum states and generalized observables: A simple proof of gleason's theorem," *Physical Review Letters*, vol. 91, Sept. 2003.

[135] C. M. Caves, C. A. Fuchs, K. K. Manne, and J. M. Renes, "Gleason-type derivations of the quantum probability rule for generalized measurements," *Foundations of Physics*, vol. 34, p. 193–209, Feb. 2004.

[136] J. Barrett, "Nonsequential positive-operator-valued measurements on entangled mixed states do not always violate a bell inequality," *Physical Review A*, vol. 65, Mar. 2002.

[137] A. Rai, M. R. Gazi, M. Banik, S. Das, and S. Kunkri, "Local simulation of singlet statistics for a restricted set of measurements," *Journal of Physics A: Mathematical and Theoretical*, vol. 45, p. 475302, Nov. 2012.

[138] K. Kraus, *States, Effects, and Operations Fundamental Notions of Quantum Theory*. Springer Berlin Heidelberg, 1983.

[139] O. Gühne and G. Tóth, "Entanglement detection," *Physics Reports*, vol. 474, p. 1–75, Apr. 2009.

[140] M. Banik, M. R. Gazi, S. Ghosh, and G. Kar, "Degree of complementarity determines the nonlocality in quantum mechanics," *Physical Review A*, vol. 87, May 2013.

[141] F. Graffitti, A. Pickston, P. Barrow, M. Proietti, D. Kundys, D. Rosset, M. Ringbauer, and A. Fedrizzi, "Measurement-device-independent verification of quantum channels," *Physical Review Letters*, vol. 124, Jan. 2020.

[142] S. Woronowicz, "Positive maps of low dimensional matrix algebras," *Reports on Mathematical Physics*, vol. 10, p. 165–183, Oct. 1976.

[143] E. Størmer, *Positive Linear Maps of Operator Algebras*. Springer Berlin Heidelberg, 2013.

[144] W. van Dam, "Implausible consequences of superstrong nonlocality," *Natural Computing*, vol. 12, p. 9–12, Nov. 2012.

[145] I. H. Deutsch, "Harnessing the power of the second quantum revolution," *PRX Quantum*, vol. 1, Nov. 2020.

[146] D. Bohm, "A suggested interpretation of the quantum theory in terms of "hidden" variables. i," *Physical Review*, vol. 85, p. 166–179, Jan. 1952.

[147] H. Everett, ""relative state" formulation of quantum mechanics," *Reviews of Modern Physics*, vol. 29, p. 454–462, July 1957.

[148] J. A. Wheeler, "Assessment of everett's "relative state" formulation of quantum theory," *Reviews of Modern Physics*, vol. 29, p. 463–465, July 1957.

[149] L. E. BALLENTINE, "The statistical interpretation of quantum mechanics," *Reviews of Modern Physics*, vol. 42, p. 358–381, Oct. 1970.

[150] R. B. Griffiths, "Consistent histories and the interpretation of quantum mechanics," *Journal of Statistical Physics*, vol. 36, p. 219–272, July 1984.

[151] M. Tegmark, "The interpretation of quantum mechanics: Many worlds or many words?," *Fortschritte der Physik*, vol. 46, p. 855–862, Nov. 1998.

[152] M. Schlosshauer, J. Kofler, and A. Zeilinger, "A snapshot of foundational attitudes toward quantum mechanics," *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, vol. 44, p. 222–230, Aug. 2013.

[153] M. Navascués and H. Wunderlich, "A glance beyond the quantum model," *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 466, p. 881–890, Nov. 2009.

[154] T. Fritz, A. Sainz, R. Augusiak, J. B. Brask, R. Chaves, A. Leverrier, and A. Acín, "Local orthogonality as a multipartite principle for quantum correlations," *Nature Communications*, vol. 4, Aug. 2013.

[155] B. Amaral, M. T. Cunha, and A. Cabello, "Exclusivity principle forbids sets of correlations larger than the quantum set," *Physical Review A*, vol. 89, Mar. 2014.

[156] A. Ahanj, S. Kunkri, A. Rai, R. Rahaman, and P. S. Joag, “Bound on hardy’s nonlocality from the principle of information causality,” *Physical Review A*, vol. 81, Mar. 2010.

[157] S. Das, M. Banik, M. R. Gazi, A. Rai, S. Kunkri, and R. Rahaman, “Bound on tri-partite hardy’s nonlocality respecting all bi-partite principles,” *Quantum Information Processing*, vol. 12, p. 3033–3041, May 2013.

[158] M. R. Gazi, M. Banik, S. Das, A. Rai, and S. Kunkri, “Macroscopic locality with equal bias reproduces with high fidelity a quantum distribution achieving the tsirelson’s bound,” *Physical Review A*, vol. 88, Nov. 2013.

[159] N. Miklin and M. Pawłowski, “Information causality without concatenation,” *Physical Review Letters*, vol. 126, June 2021.

[160] R. W. Spekkens, “The status of determinism in proofs of the impossibility of a noncontextual model of quantum theory,” *Foundations of Physics*, vol. 44, p. 1125–1155, Sept. 2014.

[161] E. S. S. Kochen, “The problem of hidden variables in quantum mechanics,” *Indiana Univ. Math. J.*, vol. 17, pp. 59–87, 1968.

[162] N. Harrigan and R. W. Spekkens, “Einstein, incompleteness, and the epistemic view of quantum states,” *Foundations of Physics*, vol. 40, p. 125–157, Jan. 2010.

Appendix A

Proof of Proposition 3

Proof. The POPTness of the state directly follows from the expression $\mathrm{Tr}[W_p(\mathbb{P}_{\hat{n}}\otimes\mathbb{P}_{\hat{m}})] = 1/4(1+p\ \hat{n}.\hat{m})$, where $\mathbb{P}_{\hat{x}} := 1/2(\mathbb{I}+\hat{x}.\sigma)$; and beyond quantumness follows from the explicit eigenvalue calculation of the operator W_p.

Our aim is to show that for certain range of the parameter p, the classical-input classical-output correlations obtained from the class of BQSs $W_p := p\Gamma[|\phi^+\rangle\langle\phi^+|] + (1-p)\mathbb{I}/4$ can be classically simulated. Given the classical inputs, the parties Alice and Bob perform some local measurement on their part of the BQS to obtained some classical outputs. The joint input-output probabilities are calculated using Born rule as the local systems are assumed to be quantum. By classically simulable we mean that the obtained correlations allow a local hidden variable (LHV) model, *i.e.* if Alice and Bob perform some measurements $A \equiv \{A_i \mid A_i \geq 0\ \&\ \sum_i A_i = \mathbb{I}\}$ and $B \equiv \{B_j \mid B_j \geq 0\ \&\ \sum_j B_j = \mathbb{I}\}$ respectively, then the joint probability distributions are factorizable.

$$P(A_i, B_j|A, B, W_p) = \int_\Lambda \omega(\lambda|W_p) P(A_i|A, \lambda) P(B_j|B, \lambda) d\lambda, \tag{A.1}$$

where $\lambda \in \Lambda$ is some shared variable (also called common cause/HV) and $\omega(\lambda|W_p)$ is a probability distribution on the HV space Λ.

Let us first consider the particular case, where measurement effects of Alice's and Bob's measurements are proportional to some rank one projection operator, *i.e.* $A_i = x_i P_i$ & $B_j = y_j Q_j$, with $0 < x_i, y_j \leq 1$. Note that, $\mathrm{Tr}\left[\Gamma[|\phi^+\rangle\langle\phi^+|]\left(P_i\otimes Q_j\right)\right] = 1/4(1+\hat{m}_i.\hat{n}_j)$, where $P_i := 1/2(\mathbb{I}+\hat{m}_i.\sigma)$ and $Q_j := 1/2(\mathbb{I}+\hat{n}_j.\sigma)$. This expression differs from $\mathrm{Tr}\left[|\psi^-\rangle\langle\psi^-|\left(P_i\otimes Q_j\right)\right] = 1/4(1-\hat{m}_i.\hat{n}_j)$ just by a negative sign, which motivates us to construct the LHV for W_p by simply modifying the LHV model known for the noisy singlet state [136].

Let the hidden variables $\lambda \in \Lambda$ (which we shall now denote by $|\lambda\rangle$) be the unit vectors of a 2-dimensional complex Hilbert space. The local responses are given by,

$$P(A_i|A,\lambda) := \langle\lambda|A_i|\lambda\rangle\,\Theta\left(\langle\lambda|P_i|\lambda\rangle-\frac{1}{2}\right)+\frac{x_i}{2}\left(1-\sum_k\langle\lambda|A_k|\lambda\rangle\,\Theta\left(\langle\lambda|P_i|\lambda\rangle-\frac{1}{2}\right)\right); \tag{A.2}$$

$$P(B_j|B,\lambda) := y_j\left(1-\langle\lambda^\perp|Q_j|\lambda^\perp\rangle\right); \tag{A.3}$$

where $\Theta(x)$ is the Heaviside step function and $|\lambda^\perp\rangle$ is the state perpendicular to $|\lambda\rangle$. It is noteworthy that the response on Alice's side is contextual, since the response of the effect A_i depends on the other effects A_k's constituting the measurement A. Substituting Eqs.(A.2) and (A.3) in Eq.(A.1) we obtain

$$\begin{aligned}P\left(A_i,B_j|A,B,W_p\right) = \int_\Lambda \omega\left(\lambda|W_p\right)\left[y_j\left(1-\langle\lambda^\perp|Q_j|\lambda^\perp\rangle\right)\right]\times\Bigg[\langle\lambda|A_i|\lambda\rangle\,\Theta\left(\langle\lambda|P_i|\lambda\rangle-\frac{1}{2}\right)\\ +\frac{x_i}{2}\left(1-\sum_k\langle\lambda|A_k|\lambda\rangle\,\Theta\left(\langle\lambda|P_i|\lambda\rangle-\frac{1}{2}\right)\right)\Bigg]d\lambda.\end{aligned} \tag{A.4}$$

To check the reproducibility condition let us define the following quantity:

$$J_{ij} := x_i y_j \int d\lambda\ \omega(\lambda|W_p)\Theta\left(\langle\lambda|P_i|\lambda\rangle-\frac{1}{2}\right)\langle\lambda|P_i|\lambda\rangle\langle\lambda^\perp|Q_j|\lambda^\perp\rangle. \tag{A.5}$$

Using Eq.(A.5), Eq.(A.4) can be written as

$$\begin{aligned}P\left(A_i,B_j|A,B,W_p\right) &= \left(-J_{ij}-\frac{1}{2}x_i y_j\int d\lambda\ \omega\left(\lambda|W_p\right)\langle\lambda^\perp|Q_j|\lambda^\perp\rangle\right)+\left(\frac{x_i}{2}\sum_k J_{kj}\right)\\ &\quad+\left(y_j\sum_l J_{il}\right)+\left(\frac{x_i y_j}{2}\int d\lambda\ \omega\left(\lambda|W_p\right)\right)-\left(\frac{x_i y_j}{2}\sum_{kl}J_{kl}\right)\\ &=\frac{x_i y_j}{2}\left(\int d\lambda\ \omega\left(\lambda|W_p\right)-\int d\lambda\ \omega\left(\lambda|W_p\right)\langle\lambda^\perp|Q_j|\lambda^\perp\rangle\right)\\ &\quad+\left(-J_{ij}+\frac{x_i}{2}\sum_k J_{kj}\right)+\left(y_j\sum_l J_{il}-\frac{x_i y_j}{2}\sum_{kl}J_{kl}\right).\end{aligned} \tag{A.6}$$

The quantity $c \equiv \int d\lambda\ \omega\left(\lambda|W_p\right)\langle\lambda^\perp|Q_j|\lambda^\perp\rangle$ is invariant under the change of Q_j. Now, $\sum_j y_j c = \sum_j y_j \int d\lambda\ \omega\left(\lambda|W_p\right)\langle\lambda^\perp|Q_j|\lambda^\perp\rangle = \int d\lambda\ \omega\left(\lambda|W_p\right) = 1$. Thus $c = 1/2$;

which yields,

$$P\left(A_i,B_j|A,B,W_p\right)=\frac{x_iy_j}{4}+\left(-J_{ij}+\frac{x_i}{2}\sum_k J_{kj}\right)+\left(y_j\sum_l J_{il}-\frac{x_iy_j}{2}\sum_{kl}J_{kl}\right). \tag{A.7}$$

In order to evaluate J_{ij} we write $|\lambda\rangle = z_0|0\rangle + z_1|1\rangle$ where $\{|0\rangle,|1\rangle\}$ is an orthonormal basis for $\mathbb{C}^2$. Let $z_\nu = r_\nu e^{i\theta_\nu}$ for $\nu \in \{0,1\}$. We choose $|0\rangle$ to be such that $|0\rangle\langle 0| = P_i$. Writing $u_\nu = r_\nu^2$ and $Q_j = |q_j\rangle\langle q_j|$ and using the fact $\langle\lambda^\perp|q_j\rangle\langle q_j|\lambda^\perp\rangle = \langle\lambda|q_j^\perp\rangle\langle q_j^\perp|\lambda\rangle$, we get

$$\begin{aligned}J_{ij} &= x_iy_j\int d\lambda\ \omega(\lambda|W_p)\,\Theta(\langle\lambda|P_i|\lambda\rangle-1/2)\,\langle\lambda|P_i|\lambda\rangle\,\langle\lambda|q_j^\perp\rangle\langle q_j^\perp|\lambda\rangle\\ &=\frac{1}{N}x_iy_j\sum_{\nu=0}^{1}|\langle q_j^\perp|\nu\rangle|^2\int_{1/2}^{1}du_0\int_0^1 du_1\ \delta(u_0+u_1-1)\ u_0u_\nu = x_iy_j\sum_{\nu=0}^{1}|\langle q_j^\perp|\nu\rangle|^2\,J_\nu,\end{aligned} \tag{A.8}$$

where we have assumed $\omega(\lambda|W_p)$ to be a uniform distribution over Λ and

$$N := \int_0^1 du_0\int_0^1 du_1\ \delta(u_0+u_1-1), \tag{A.9}$$

$$J_\nu := \frac{1}{N}\int_{1/2}^{1}du_0\int_0^1 du_1\ \delta(u_0+u_1-1)\ u_0u_\nu. \tag{A.10}$$

Defining $\tilde{J}\equiv\frac{1}{N}\int_{1/2}^{1}du_0\int_0^1du_1\ \delta(u_0+u_1-1)\ u_0$ and using $u_0+u_1=1$, we can write $J_1=\tilde{J}-J_0$. Using normalization condition for $|q_j^\perp\rangle$ and the fact $|p_i\rangle=|0\rangle$ we get $|\langle q_j^\perp|1\rangle|^2 = 1-|\langle q_j^\perp|p_i\rangle|^2$, which thus yields

$$J_{ij}=x_iy_j\left[\left(\tilde{J}-J_0\right)+\left(2J_0-\tilde{J}\right)|\langle q_j^\perp|p_i\rangle|^2\right]. \tag{A.11}$$

Substituting Eq.(A.11) in Eq.(A.7) we get,

$$P\left(A_i,B_j|A,B,W_p\right)=x_iy_j\left(\frac{1+4J_0-2\tilde{J}}{4}-(2J_0-\tilde{J})|\langle q_j^\perp|p_i\rangle|^2\right) \tag{A.12}$$

Straightforward integration yields $J_0=\frac{7}{24}$ and $\tilde{J}=\frac{3}{8}$, which further implies

$$\begin{aligned}P\left(A_i,B_j|A,B,W_p\right)&=\frac{x_iy_j}{48}\left(17-10|\langle q_j^\perp|p_i\rangle|^2\right)=\frac{x_iy_j}{48}\left(17-10\,\mathrm{Tr}\left(P_iQ_j^\perp\right)\right)\\&=x_iy_j\frac{1}{4}\left(1+\frac{5}{12}\hat{m}_i\cdot\hat{n}_j\right).\end{aligned} \tag{A.13}$$

Again, from the Born rule we have

$$P\left(A_i, B_j|A, B, W_p\right) = \mathrm{Tr}\left[(A_i \otimes B_j)(W_p)\right] = x_i y_j \frac{1}{4}\left(1 + p\ \hat{m}_i \cdot \hat{n}_j\right). \tag{A.14}$$

Therefore, for $p = \frac{5}{12}$ the BQS W_p allows a LHV model when all the effects constituting Alice's and Bob's measurements are proportional to rank one projectors. We are now left to extend this model for more general measurements (consisting more than rank one effects). This can be argued by noticing that any POVM element $\mathscr{A}$ is a Hermitian operator with $0 < \mathscr{A} \leq 1$, and hence allows spectral decomposition of the form $\mathscr{A} = \sum_i A_i$, where $A_i = x_i P_i$ are operators proportional to rank-one projectors like the ones considered above with $0 < x_i \leq 1$ and $P_i P_j = \delta_{ij} P_i$. Thus any general POV measurement can be regarded as a coarse-grained measurement of the special scenario considered above. We associate the outcome A_i of the finer measurement with the outcome $\mathscr{A}$ of the coarse-grained measurement for all values of i. Thus we have a LHV model for $W_{5/12}$.

Once the LHV model is defined for a particular state, it can be extended for a large class of states. Suppose we have a LHV model for the state σ_1. It is then possible to construct a LHV model for a state σ_2 if it can be written in the form $\sigma_2 = \sum_{ij} M_i \otimes N_j\ \sigma_1\ M_i^\dagger \otimes N_j^\dagger$, such that $\sum_i M_i^\dagger M_i = \mathbb{I}$, & $\sum_j N_j^\dagger N_j = \mathbb{I}$. For describing the LHV model of σ_2 we just need to modify the responses in the following way

$$P_{\sigma_2}(A_i|A,\lambda) := P_{\sigma_1}(A_i'|A',\lambda) \ \text{ where } \ A_i' := \sum_k M_k^\dagger A_i M_k, \tag{A.15}$$

$$P_{\sigma_2}(B_j|B,\lambda) := P_{\sigma_1}(B_j'|B',\lambda) \ \text{ where } \ B_j' := \sum_l N_l^\dagger B_j N_l. \tag{A.16}$$

If we now take $\omega(\lambda|\sigma_2) = \omega(\lambda|\sigma_1)$, the above construction will give

$$\begin{aligned} \int d\lambda\ \omega(\lambda|\sigma_2) P_{\sigma_2}(A_i|A,\lambda) P_{\sigma_2}(B_j|B,\lambda) &= \mathrm{Tr}[\left(A_i' \otimes B_j'\right)\sigma_1] \\ &= \sum_{kl} \mathrm{Tr}[\left(M_k^\dagger A_i M_k \otimes N_l^\dagger B_j N_l\right)\sigma_1] \\ &= \sum_{kl} \mathrm{Tr}[\left(A_i \otimes B_j\right)\left(M_k \otimes N_l\ \sigma_1\ M_k^\dagger \otimes N_l^\dagger\right)] \\ &= \mathrm{Tr}[\left(A_i \otimes B_j\right)\sigma_2]. \end{aligned} \tag{A.17}$$

Thus the above construction is successful in defining a valid LHV model for σ_2. It is obvious that any $W_{p'}$ can be created from W_p just by using local operations if $p' \leq p$. This implies existence of a LHV model for $W_{p \leq \frac{5}{12}}$. Therefore, any classical-input classical-output correlation obtained from W_p is classically simulable whenever $p \leq 5/12$. On the other hand, as discussed in the manuscript, W_p is BQS for $p > 1/3$. This proves the claim of Proposition **1**. □

Remark: Motivated from the LHV model constructed in [21], it can be further shown that for W_p a classical model exists for $p \leq 1/2$ whenever Alice's and Bob measurements are limited to projective measurement only. In this case also λ's are given by unit vectors of 2-dimensional complex Hilbert space and $\omega(\lambda|W_p)$ is taken to be uniform distribution. Using spherical polar coordinates we can denote the HVs as $\hat{\lambda} = \sin(\theta)\cos(\phi)\hat{\imath} + \sin(\theta)\sin(\phi)\hat{\jmath} + \cos(\theta)\hat{k}$ and $\omega(\lambda|W_p)d\lambda = \frac{1}{4\pi}\sin(\theta)d\theta d\phi$. Alice's and Bob's response are given by

$$P(P_i|A,\lambda) = \cos^2(\alpha_1/2) = \frac{1+\cos(\alpha_1)}{2}; \tag{A.18}$$

$$P(Q_j|B,\lambda) = 1 \quad \text{if} \quad 2\cos^2(\alpha_2/2) < 1; \tag{A.19}$$

$$= 0 \quad \text{if} \quad 2\cos^2(\alpha_2/2) > 1. \tag{A.20}$$

Here α_1 is the angle between the block vector of P_i and $-\hat{\lambda}$, and α_2 is the angle between the block vector of Q_j and $\hat{\lambda}$. Without any loss of generality we can consider $P_i = 1/2(\mathbb{I} + \hat{m}_i.\sigma)$ and $Q_j = 1/2(\mathbb{I} + \hat{n}_j.\sigma)$ with $\hat{m}_i = (\sin\ x, 0, \cos\ x)$ and $\hat{n}_j = (0,0,1)$. This implies $P(Q_j|B,\lambda) = 1$ for $\frac{\pi}{2} < \alpha_2 < \frac{3\pi}{2}$ and accordingly we have non zero contribution in the integral for $\frac{\pi}{2} < \theta < \pi$. Also we get $\cos(\alpha_1) = -\sin(x)\sin(\theta)\cos(\phi) - \cos(x)\cos(\theta)$. Therefore, we have

$$\begin{aligned} P\left(P_i, Q_j|A,B,W_p\right) &= \int_{\theta=\frac{\pi}{2}}^{\theta=\pi} \int_{\phi=0}^{\phi=2\pi} \frac{1}{4\pi}\sin(\theta)\frac{1}{2}\left[1 - \sin(x)\sin(\theta)\cos(\phi) - \cos(x)\cos(\theta)\right]\, d\theta\, d\phi \\ &= \frac{1}{4} + \frac{\cos\ x}{8} = \frac{1}{4}(1 + \frac{1}{2}\hat{m}_i \cdot \hat{n}_j), \end{aligned} \tag{A.21}$$

which is same as the Born probability obtained from the state W_p for $p = 1/2$. It is not difficult to see that the model can be extended for any values of $p \leq 1/2$.

Appendix B

Composing POPT sates

In the Semiquantum Bell scenario, the Prover performs local measurements $M_{A_iA_i^o} = \{\pi^{a_i}_{A_iA_i^o}\}_{a_i}$ on the i^{th} subsystem. In the locally quantum no-signalling framework, the composite multipartite state is given by a functional $\mu_{A_1A_1^o \cdots A_nA_n^o}$ which, invoking unentangled Gleason's theorem, corresponds to a POPT state $Z_{A_1A_1^o \cdots A_nA_n^o}$. It is in general not true that $Z_{A_1A_1^o \cdots A_nA_n^o}$ is given by the tensor product of the individual states. If the states held by the Verifiers are pure and unentangled, then one can show that $Z_{A_1A_1^o \cdots A_nA_n^o} = \otimes_i \psi^{s_i}_{A_i^o} \otimes W_{A_1 \cdots A_n}$. To begin with, it is easy to check that $\otimes_i \psi^{s_i}_{A_i^o} \otimes W_{A_1 \cdots A_n}$ is a valid POPT state, as shown below:

Proposition 6. *For every POPT state $W_{A_1 \cdots A_n}$, the tensor product state $W_{A_1 \cdots A_n} \otimes \psi^{s_1}_{A_1^o} \otimes \cdots \otimes \psi^{s_n}_{A_n^o}$ is also a POPT state.*

Proof. For any set of local POVMs $\pi_{A_1A_1^o} \otimes \cdots \otimes \pi_{A_nA_n^o}$ acting on the tensor product state $W_{A_1 \cdots A_n} \otimes \psi^{s_1}_{A_1^o} \otimes \cdots \otimes \psi^{s_n}_{A_n^o}$, we have,

$$\begin{aligned}\mathrm{Tr}\left[\left(\pi_{A_1A_1^o} \otimes \cdots \otimes \pi_{A_nA_n^o}\right)\left(W_{A_1 \cdots A_n} \otimes \psi^{s_1}_{A_1^o} \otimes \cdots \otimes \psi^{s_n}_{A_n^o}\right)\right] &= \mathrm{Tr}\left[\left(\pi_{A_1} \otimes \cdots \otimes \pi_{A_n}\right)\left(W_{A_1 \cdots A_n}\right)\right] \\ &\geq 0\end{aligned}$$

where, $\pi_{A_j} := \mathrm{Tr}_{A_j^o}\left[\left(\mathbb{I}_{A_j} \otimes \psi^{s_j}_{A_j^o}\right)\left(\pi_{A_jA_j^o}\right)\right]$ is a positive operator for every POVM element $\pi_{A_jA_j^o}$.

The final inequality follows from the fact that $W_{A_1 \cdots A_n}$ is a POPT state and gives positive probabilities for local measurements. Therefore, $W_{A_1 \cdots A_n} \otimes \psi^{s_1}_{A_1^o} \otimes \cdots \otimes \psi^{s_n}_{A_n^o}$ produces a positive probability for all local measurements. Hence, it is a valid POPT state. □

We next show that subsystems of POPT states are obtained by the partial trace operation. In what follows, we restrict the analysis to bipartite scenarios for notational convenience. The generalization to multipartite scenarios is obvious. For any local measurements $M_{AA^o} = \{\pi^a_{AA^o}\}_a$ and $M_{BB^o} = \{\pi^b_{BB^o}\}_b$ we have,

$$\mu_{AA^oBB^o}(\pi^a_{AA^o}, \pi^b_{BB^o}) = \text{Tr}[Z_{AA^oBB^o}(\pi^a_{AA^o} \otimes \pi^b_{BB^o})]$$

Let $\pi^a_{AA^o} = \pi^\alpha_A \otimes \pi^{\alpha^o}_{A^o}$ and $\pi^b_{BB^o} = \pi^\beta_B \otimes \pi^{\beta^o}_{B^o}$ with $\sum_\alpha \pi^\alpha_A = \mathbb{I}_A, \sum_{\alpha^o} \pi^{\alpha^o}_{A^o} = \mathbb{I}_{A^o}, \sum_\beta \pi^\beta_B = \mathbb{I}_B, \sum_{\beta^o} \pi^{\beta^o}_{B^o} = \mathbb{I}_{B^o}$.

$$\mu_{AA^oBB^o}(\pi^\alpha_A \otimes \pi^{\alpha^o}_{A^o}, \pi^\beta_B \otimes \pi^{\beta^o}_{B^o}) = \text{Tr}[Z_{AA^oBB^o}(\pi^\alpha_A \otimes \pi^{\alpha^o}_{A^o} \otimes \pi^\beta_B \otimes \pi^{\beta^o}_{B^o})] \tag{B.1}$$

Summing over α^o and β^o,

$$\mu_{AB}(\pi^\alpha_A, \pi^\beta_B) = \text{Tr}[W_{AB}(\pi^\alpha_A \otimes \pi^\beta_B)] = \text{Tr}[Z_{AA^oBB^o}(\pi^\alpha_A \otimes \mathbb{I}_{A^o} \otimes \pi^\beta_B \otimes \mathbb{I}_{B^o})] \tag{B.2}$$

Where, we have used $\sum_{\alpha^o,\beta^o} \mu_{AA^oBB^o}(\pi^\alpha_A \otimes \pi^{\alpha^o}_{A^o}, \pi^\beta_B \otimes \pi^{\beta^o}_{B^o}) = \sum_{\alpha^o,\beta^o} p(\alpha, \alpha^o, \beta, \beta^o | M_{AA^o}M_{BB^o}) = p(\alpha, \beta | M_A M_B) = \mu_{AB}(\pi^\alpha_A, \pi^\beta_B)$. Since equation Eq. (B.2) is true for all POVMs π^α_A and π^β_B, we get,

$$\text{Tr}_{A^oB^o}(Z_{AA^oBB^o}) = W_{AB} \tag{B.3}$$

If we sum over β and β^o in Eq. (B.1), we get

$$\mu_{AA^o}(\pi^\alpha_A, \pi^{\alpha^o}_{A^o}) = \text{Tr}[\text{Tr}_{BB^o}(Z_{AA^oBB^o})(\pi^\alpha_A \otimes \pi^{\alpha^o}_{A^o})] \tag{B.4}$$

$\text{Tr}_{BB^o}(Z_{AA^oBB^o})$ is a Positive Operator since $Z_{AA^oBB^o}$ must be a POPT in the AA^o/BB^o cut. If we now sum Eq. (B.4) over α we end up with,

$$\mu_{A^o}(\pi^{\alpha^o}_{A^o}) = \text{Tr}(\psi_{A^o}\pi^{\alpha^o}_{A^o}) = \text{Tr}[\text{Tr}_{BB^o}(Z_{AA^oBB^o})(\mathbb{I}_A \otimes \pi^{\alpha^o}_{A^o})] \tag{B.5}$$

where ψ_{A^o} is the state given the Prover by the Verifier Alice. Since Eq. (B.5) is true for all $\pi^{\alpha^o}_{A^o}$, we have,

$$\text{Tr}_A(\text{Tr}_{BB^o}(Z_{AA^oBB^o})) = \psi_{A^o} \tag{B.6}$$

$$\text{Tr}_{A^o}(\text{Tr}_{BB^o}(Z_{AA^oBB^o})) = \text{Tr}_B(W_{AB}) \quad \text{[From Eq. (B.3)]} \tag{B.7}$$

Along with the fact that $\text{Tr}_{BB^o}(Z_{AA^oBB^o})$ and $\text{Tr}_B(W_{AB})$ are Positive operators, and that ψ_{A^o} is a pure quantum state, equations (B.6) and (B.7) imply,

$$\text{Tr}_{BB^o}(Z_{AA^oBB^o}) = \text{Tr}_B(W_{AB}) \otimes \psi_{A^o} \tag{B.8}$$

A similar argument on Bob's side yeilds,

$$\mathrm{Tr}_{AA^o}(Z_{AA^oBB^o}) = \mathrm{Tr}_A(W_{AB}) \otimes \psi_{B^o} \tag{B.9}$$

Note that it is essential for the states ψ_{A^o} and ψ_{B^o} to be pure in order for equations (B.8) and (B.9) to hold. From equations (B.3), (B.8), and (B.9) we conclude that

$$Z_{AA^oBB^o} = \psi_{A^o} \otimes W_{AB} \otimes \psi_{B_o}$$

which is the required expression.

We mention here that $Z_{AA^oBB^o}$ is not always given by the tensor product of individual states. The tensor product of a BQS and an entangled state may not necessarily be a valid POPT, as we show below.

Proposition 7. *[Barnum et al.; arXiv:quant-ph/0507108] For every BQS $W_{A_1\cdots A_n}$, there exists an entangled state $\rho^{s_1\cdots s_n}_{A_1^o\cdots A_n^o}$ such that the tensor product state $W_{A_1\cdots A_n} \otimes \rho^{s_1\cdots s_n}_{A_1^o\cdots A_n^o}$ is not a POPT sate.*

Proof. Given a BQS $W_{A_1\cdots A_n}$, let $\chi^{s_1\cdots s_n}_{A_1^o\cdots A_n^o}$ be the eigen-projector corresponding to the negative eigenvalue in the spectral decomposition of $W_{A_1\cdots A_n}$. From Eq. (D.4), $\chi^{\mathrm{T}}_{A_1\cdots A_n} = \sum_{s_1,\cdots,s_n} \beta_{s_1\cdots s_n} \otimes_{i=1}^{n} \psi^{s_i}_{A_i}$. For the local measurements $\left\{P^+_{A_iA_i^o}, \mathbb{I}_{A_iA_i^o} - P^+_{A_iA_i^o}\right\}$ defined in Theorem 2, the probability of the occurrence of outcome $P^+_{A_1A_1^o} \otimes \cdots \otimes P^+_{A_nA_n^o}$ for the tensor product state $W_{A_1\cdots A_n} \otimes (\chi^{s_1\cdots s_n}_{A_1^o\cdots A_n^o})^{\mathrm{T}}$ is,

$$\begin{aligned}
&\mathrm{Tr}\left[\left(P^+_{A_1A_1^o} \otimes \cdots \otimes P^+_{A_nA_n^o}\right)\left(W_{A_1\cdots A_n} \otimes (\chi^{s_1\cdots s_n}_{A_1^o\cdots A_n^o})^{\mathrm{T}}\right)\right] \\
&= \sum_{s_1,\cdots,s_n} \beta_{s_1\cdots s_n} \times \mathrm{Tr}\left[\left(P^+_{A_1A_1^o} \otimes \cdots \otimes P^+_{A_nA_n^o}\right)\left(W_{A_1\cdots A_n} \otimes \psi^{s_1}_{A_1^o} \otimes \cdots \otimes \psi^{s_n}_{A_n^o}\right)\right] \\
&< 0
\end{aligned} \tag{B.10}$$

The final inequality follows from the fact that Eq.(B.10) is the same as Eq.(C.1), where it is shown that the RHS is negative. Since $W_{A_1\cdots A_n} \otimes (\chi^{s_1\cdots s_n}_{A_1^o\cdots A_n^o})^{\mathrm{T}}$ gives negative probabilities upon local measurement, it is not a valid POPT state. Therefore, $(\chi^{s_1\cdots s_n}_{A_1^o\cdots A_n^o})^{\mathrm{T}}$ is the required state $\rho^{s_1\cdots s_n}_{A_1^o\cdots A_n^o}$ for the proof of our Proposition. □

The above result does *not* mean that the composition of an entangled state and BQS is not defined. Rather, the result must be interpreted to mean that the global state $Z_{AA^oBB^o}$ obtained by composing an entangled state $\rho_{A^oB^o}$ and a BQS

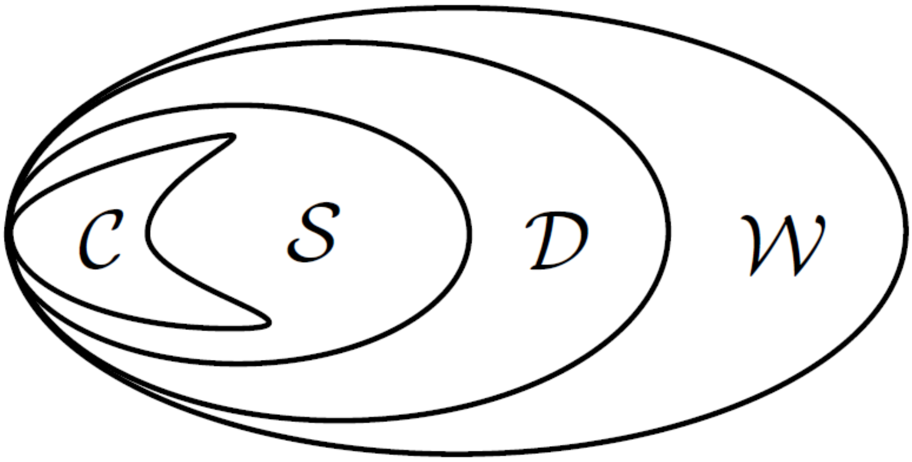


FIG. B.1 Tensoring of POPT states. $\mathscr{W}$ is the set of all POPT states, $\mathscr{D}$ is the set of density operators, $\mathscr{S}$ is the set of separable states, and $\mathscr{C}$ is the set of all classical-classical states having zero discord. While all the states in $\mathscr{W}$ allow local quantum description, interesting scenarios arise when tensoring of such states is considered [see Propositions (6) and (7)]. While Barnum *et al.* considered the restricted scenario in which states were of the form $w \otimes c$ with $w \in \mathscr{W}$ and $c \in \mathscr{C}$, we have considered more general states of the form $w \otimes \phi$ with $w \in \mathscr{W}$ and $\phi \in \mathscr{S}$.

W_{AB} is not necessarily given by the tensor product of the individual states. We refer to Proposition 6.2 of [129] for further details.

Appendix C

Proof of Theorem 9

Proof. This proof is a straightforward generalization of the proof of Theorem **1**. For every BQS $W_{A_1\cdots A_n} \in \mathscr{W}(\bigotimes_{i=1}^{n} \mathscr{H}_{A_i})$ there exists an entangled state $\chi_{A_1\cdots A_n} \in \mathscr{D}(\bigotimes_{i=1}^{n} \mathscr{H}_{A_i})$ such that $\mathrm{Tr}[W_{A_1\cdots A_n}\chi_{A_1\cdots A_n}] < 0$, whereas $\mathrm{Tr}[\sigma_{A_1\cdots A_n}\chi_{A_1\cdots A_n}] \geq 0, \forall\, \sigma_{A_1\cdots A_n} \in \mathscr{D}(\bigotimes_{i=1}^{n} \mathscr{H}_{A_i})$ [139]. The state allows non-unique decomposition of the form

$$\chi_{A_1\cdots A_n} = \sum_{s_1,\cdots,s_n} \beta_{s_1\cdots s_n} \bigotimes_{i=1}^{n} \psi_{A_i}^{s_i\,\mathrm{T}}, \text{ where } \psi_{A_i}^{s_i} \in \mathscr{D}(\mathscr{H}_{A_i}) \;\;\&\;\; \beta_{s_1\cdots s_n} \in \mathbb{R}.$$

In the semiquantum game, referee sends the quantum inputs $\psi_{A_i^o}^{s_i}$ to the i^{th} party who has to produce binary outputs $\in \{0,1\}$. Their average payoff will be calculated as

$$\mathscr{I}_{\mathbb{G}_{sq}^{\chi}} := \sum_{s_1,\cdots,s_n} \beta_{s_1\cdots s_n} \times p\left(1\cdots 1|\psi_{A_1^o}^{s_1}\cdots\psi_{A_n^o}^{s_N}\right).$$

Given the BQS $W_{A_1\cdots A_n}$, the i^{th} party performs the measurement $\left\{P^{+}_{A_iA_i^o},\, \mathbb{I}_{A_iA_i^o} - P^{+}_{A_iA_i^o}\right\}$ on her part of the shared BQS and the quantum input $\psi_{A_i^o}^{s_i}$ received from the referee. Here $P^{+}_{A_iA_i^o} := |\phi^{+}\rangle_{A_iA_i^o}\langle\phi^{+}|$ with $|\phi^{+}\rangle_{A_iA_i^o} := \frac{1}{\sqrt{d_{A_i}}}\sum_{i=0}^{d_{A_i}-1}|ii\rangle$ and $P^{+}_{A_iA_i^o}$ corresponds to the output 1. The average payoff turns out to be,

$$\mathscr{I}_{\mathbb{G}_{sq}^{\chi}}(W_{A_1\cdots A_n}) = \sum_{s_1,\cdots,s_n} \beta_{s_1\cdots s_n} \times \mathrm{Tr}\left[\left(P^{+}_{A_1A_1^o}\otimes\cdots\otimes P^{+}_{A_nA_n^o}\right)\left(W_{A_1\cdots A_n}\otimes\psi_{A_1^o}^{s_1}\otimes\cdots\otimes\psi_{A_n^o}^{s_n}\right)\right] \tag{C.1}$$

$$= \sum_{s_1,\cdots,s_n} \beta_{s_1\cdots s_n} \times \mathrm{Tr}\left[(R_{A_1}\otimes\cdots\otimes R_{A_n})\,W_{A_1\cdots A_n}\right],$$

where R_{A_i} is the effective POVM acting on the i^{th} party's part of $W_{A_1\cdots A_n}$, and is given by $R_{A_i} := \mathrm{Tr}_{A_i^o}\left[P^+_{A_iA_i^o}\left(\mathbb{I}_{A_i}\otimes\psi^{s_i}_{A_i^o}\right)\right] = \frac{1}{d_{A_i}}\psi^{s_i\,\mathrm{T}}_{A_i}$. Therefore, we have,

$$\begin{aligned}
\mathscr{I}_{\mathbb{G}^{\chi}_{sq}}\left(W_{A_1\cdots A_n}\right) &= \prod_{i=1}^{n} d_{A_i}^{-1}\sum_{s_1,\cdots,s_n}\beta_{s_1\cdots s_n}\times\mathrm{Tr}\left[\left(\bigotimes_{i=1}^{n}\psi^{s_i\,\mathrm{T}}_{A_i}\right)W_{A_1\cdots A_n}\right]\\
&= \prod_{i=1}^{n} d_{A_i}^{-1}\,\mathrm{Tr}\left[\left(\sum_{s_1,\cdots,s_n}\beta_{s_1\cdots s_n}\bigotimes_{i=1}^{n}\psi^{s_i\,\mathrm{T}}_{A_i}\right)W_{A_1\cdots A_n}\right]\\
&= \prod_{i=1}^{n} d_{A_i}^{-1}\,\mathrm{Tr}\left[\chi_{A_1\cdots A_n}W_{A_1\cdots A_n}\right] < 0.
\end{aligned}$$

We will now calculate the payoff for an arbitrary quantum strategy. Given a quantum state $\rho_{A_1\cdots A_N}$ let the i^{th} party perform the measurement $M_{A_iA_i^o}\equiv\{\pi^{a_i}_{A_iA_i^o}\}$ on her respective joint system, where $a_i\in\{0,1\}$. The average payoff turns out to be

$$\begin{aligned}
\mathscr{I}_{\mathbb{G}^{\chi}_{sq}}\left(\rho_{A_1\cdots A_n}\right) &= \sum_{s_1,\cdots,s_n}\beta_{s_1\cdots s_n}\times\mathrm{Tr}\left[\left(\pi^1_{A_1A_1^o}\otimes\cdots\otimes\pi^1_{A_nA_n^o}\right)\left(\rho_{A_1\cdots A_n}\otimes\psi^{s_1}_{A_1^o}\otimes\cdots\otimes\psi^{s_n}_{A_n^o}\right)\right]\\
&= \sum_{s_1,\cdots,s_n}\beta_{s_1\cdots s_n}\times\mathrm{Tr}\left[R_{A_1^o\cdots A_n^o}\left(\bigotimes_{i=1}^{n}\psi^{s_i}_{A_i^o}\right)\right],
\end{aligned}$$

where, $R_{A_1^o\cdots A_n^o} := \mathrm{Tr}_{A_1\cdots A_n}\left[\left(\pi^1_{A_1A_1^o}\otimes\cdots\otimes\pi^1_{A_nA_n^o}\right)\left(\rho_{A_1\cdots A_n}\otimes\mathbb{I}_{A_1^o\cdots A_n^o}\right)\right]$ is a positive semidefinite operator, *i.e.* $R_{A_1^o\cdots A_n^o}\in\mathscr{E}\left(\bigotimes_{i=1}^{n}\mathscr{H}_{A_i^o}\right)$. Linearity of trace further yields,

$$\mathscr{I}_{\mathbb{G}^{\chi}_{sq}}\left(\rho_{A_1\cdots A_N}\right) = \mathrm{Tr}\left[R_{A_1^o\cdots A_n^o}\left(\sum_{s_1,\cdots,s_n}\beta_{s_1\cdots s_N}\bigotimes_{i=1}^{n}\psi^{s_i}_{A_i^o}\right)\right] = \mathrm{Tr}\left[R_{A_1^o\cdots A_n^o}\,\chi^{\mathrm{T}}_{A_1^o\cdots A_n^o}\right]\geq 0. \quad \text{(C.2)}$$

The last inequality follows from the fact that $\chi^{\mathrm{T}}_{A_1^o\cdots A_n^o}\in\mathscr{D}\left(\bigotimes_{i=1}^{n}\mathscr{H}_{A_i^o}\right)$, and this completes the proof. □

Appendix D

Explicit construction of semiquantum game

Special case: Here we first construct a semiquantum game for the BQSs of form $W_p := p\Gamma[|\phi^+\rangle\langle\phi^+|] + (1-p)\mathbb{I}/4$. Clearly W_p corresponds to a BQS if and only if $1/3 < p \leq 1$. The entangled state $|\psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2} \in \mathbb{C}^2 \otimes \mathbb{C}^2$ acts as a (beyond quantum) witness for this class of states. This evidently follows from the expression:

$$W_p = \frac{p}{2}\left[|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| + |\psi^+\rangle\langle\psi^+| - |\psi^-\rangle\langle\psi^-|\right] + \frac{1-p}{4}\mathbb{I}. \tag{D.1}$$

Now the state $|\psi^-\rangle$ allows the following decomposition:

$$|\psi^-\rangle\langle\psi^-| = \frac{1}{2}\Big[P_z^{\mathrm{T}}P_{\bar{z}}^{\mathrm{T}} + P_{\bar{z}}^{\mathrm{T}}P_z^{\mathrm{T}} - \frac{1}{2}\big(P_x^{\mathrm{T}}P_x^{\mathrm{T}} - P_{\bar{x}}^{\mathrm{T}}P_x^{\mathrm{T}} - P_x^{\mathrm{T}}P_{\bar{x}}^{\mathrm{T}} + P_{\bar{x}}^{\mathrm{T}}P_{\bar{x}}^{\mathrm{T}} \\ + P_y^{\mathrm{T}}P_y^{\mathrm{T}} - P_{\bar{y}}^{\mathrm{T}}P_y^{\mathrm{T}} - P_y^{\mathrm{T}}P_{\bar{y}}^{\mathrm{T}} + P_{\bar{y}}^{\mathrm{T}}P_{\bar{y}}^{\mathrm{T}}\big)\Big]; \tag{D.2}$$

where, $P_iP_j := P_i \otimes P_j$ with P_i being the projector onto the up eigenstate of σ_i for $i \in \{x,y,z\}$ and it is the projector onto the down eigenstate for $i \in \{\bar{x},\bar{y},\bar{z}\}$. This immediately leads us to the required semiquantum game $\mathbb{G}_{sq}$. In each run of the game, referee randomly choose the states $\psi^s_{A^o} = P_s$ and $\psi^t_{B^o} = P_t$ and respectively sends them to Alice and Bob without revealing the indices s and t, where $s,t \in \{x,y,z,\bar{x},\bar{y},\bar{z}\}$. Alice and Bob needs to return classical output 1 to the referee and the average payoff will be calculated as $\mathscr{I}_{\mathbb{G}_{sq}} := \sum_{s,t}\beta_{s,t}\,p(11|\psi^s_{A^o}\psi^t_{B^o})$, where

$$\beta_{x,x} = \beta_{\bar{x},\bar{x}} = \beta_{y,y} = \beta_{\bar{y},\bar{y}} = -\beta_{\bar{x},x} = -\beta_{x,\bar{x}} = -\beta_{\bar{y},y} = -\beta_{y,\bar{y}} = \frac{1}{4},$$
$$\beta_{z,\bar{z}} = \beta_{\bar{z},z} = \frac{1}{2} \text{ and } \beta_{s,t} = 0 \text{ otherwise.}$$

The winning condition demands Alice and Bob to generate a negative payoff. If Alice performs the measurement $\{P^{+}_{AA^o}, \mathbb{I}_{AA^o} - P^{+}_{AA^o}\}$ on her part of the shared state $W_{p>1/3}$ and the quantum input $\psi^{s}_{A^o}$ received from the referee and if Bob also performs the same measurement $\{P^{+}_{BB^o}, \mathbb{I}_{BB^o} - P^{+}_{BB^o}\}$ and send the outcome 1 when the projector $P^{+}_{AA^o}/P^{+}_{BB^o}$ click, then we have

$$\mathscr{I}_{\mathbb{G}_{sq}}(W_p) = \frac{1}{d_B d_A}\mathrm{Tr}\left[\chi_{AB} W_p\right] = \frac{1}{4}\mathrm{Tr}\left[|\psi^-\rangle\langle\psi^-|W_p\right] = \frac{1-3p}{16} < 0 \text{ whenever } p > 1/3.$$

On the other hand, for every quantum strategy $\mathscr{I}_{\mathbb{G}_{sq}}(\rho) \geq 0$. It should be noted that the decomposition in Eq.(D.2) is not a unique. Considering a different decomposition it is possible to come up with a different semiquantum game. For instance, one has $|\psi^-\rangle\langle\psi^-| = \sum_{a,b=1}^{4} \beta_{ab}\psi^a \otimes \psi^b$ where $\{\beta_{ab}\}$ written in a matrix form are given by

$$[\beta_{ab}] := \begin{bmatrix} -15/64 & 17/64 & 1/2 & -1/32 \\ 17/64 & -15/64 & 1/2 & -1/32 \\ 1/2 & 1/2 & -1 & 0 \\ -1/32 & -1/32 & 0 & 1/16 \end{bmatrix},$$

and $\psi^1 := P_z$, $\psi^2 := P_{\bar{z}}$, $\psi^3 := P_x$ and $\psi^4 := P_y$.

General Case: We now provide an explicit procedure to construct a semiquantum game for any BQS $W_{A_1\cdots A_n} \in \mathscr{W}(\otimes_i \mathscr{H}_{A_i})$.

First note that for a d-dimensional Hilbert space $\mathbb{C}^d$ given an orthonormal basis $\{|a\rangle\}_{a=0}^{d-1} \subset \mathbb{C}^d$, one can construct a non-orthogonal operator basis ($\mathscr{B}^{proj}$) of Projectors from the orthogonal operator (computational) basis ($\mathscr{B}^{comp}$) as follows:

$$\mathscr{L}\left(\mathbb{C}^d\right) \supset \mathscr{B}^{comp} := \{|a\rangle\langle b|\}_{a,b=0}^{d-1};$$
$$\mathscr{L}\left(\mathbb{C}^d\right) \supset \mathscr{B}^{proj} := \{|a\rangle\langle a|\}_{a=0}^{d-1} \bigcup \{P_1^{a,b}, P_2^{a,b}\}_{a,b=0}^{d-1}, \quad a < b;$$

where $P_1^{a,b} := \frac{1}{2}\left(|a\rangle\langle a| + |b\rangle\langle b| + |a\rangle\langle b| + |b\rangle\langle a|\right)$, & $P_2^{a,b} := \frac{1}{2}\left(|a\rangle\langle a| + |b\rangle\langle b| + i|a\rangle\langle b| - i|b\rangle\langle a|\right)$. Notice that $\mathscr{B}^{proj}$ has d^2 linearly independent projectors with d number of them common to $\mathscr{B}^{comp}$. If an operator is known in the $\mathscr{B}^{comp}$ basis then it can be easily written in the $\mathscr{B}^{proj}$ basis by making the following substitution:

$$|a\rangle\langle b| = \begin{cases} P_1^{a,b} - iP_2^{a,b} - \frac{1-i}{2}|a\rangle\langle a| - \frac{1-i}{2}|b\rangle\langle b|, & a < b; \\ \\ P_1^{b,a} + iP_2^{b,a} - \frac{1+i}{2}|a\rangle\langle a| - \frac{1+i}{2}|b\rangle\langle b|, & b < a. \end{cases} \tag{D.3}$$

Now, given an arbitrary beyond quantum state $W_{A_1\cdots A_n}$, a semi quantum game can be constructed by mimicking the following steps:

S1: Write down the spectral decomposition of $W_{A_1\cdots A_n}$. Hermiticity of $W_{A_1\cdots A_n}$ guarantees that the eigenvalues are real. Since $W_{A_1\cdots A_n}$ is a BQS, it has least one negative eigenvalue with entangled eigen-projector. Let the eigen-projector corresponding to a negative eigenvalue ($\lambda < 0$) be $\chi_{A_1\cdots A_n}$. Clearly,

$$\mathrm{Tr}[W_{A_1\cdots A_n}\chi_{A_1\cdots A_n}] = \lambda < 0,$$
$$\mathrm{Tr}[\sigma_{A_1\cdots A_n}\chi_{A_1\cdots A_n}] \geq 0, \ \forall \ \sigma_{A_1\cdots A_n} \in \mathscr{D}(\bigotimes_{i=1}^{n}\mathscr{H}_{A_i}).$$

S2: Expand $\chi_{A_1\cdots A_n}$ in the computational basis:

$$\chi_{A_1\cdots A_n} = \sum_{a_i^1,\cdots,a_i^n,a_j^1,\cdots,a_j^n} \alpha_{a_i^1\cdots a_i^n a_j^1\cdots a_j^n} |a_i^1\rangle_{A_1}\langle a_j^1| \otimes \cdots \otimes |a_i^n\rangle_{A_n}\langle a_j^n|.$$

Using Eq. (D.3) we can write this as,

$$\chi_{A_1\cdots A_n} = \sum_{s_1,\cdots,s_n} \beta_{s_1\cdots s_n} \bigotimes_{i=1}^{n} \phi_{A_i}^{s_i},$$

where, $\phi_{A_i}^{s_i} \in \mathscr{B}_{A_i}^{proj}$. Since $\chi_{A_1\cdots A_n}$ is Hermitian and $\phi_{A_i}^{s_i}$ are linearly independent, all the $\beta_{s_1\cdots s_n}$ are real. Let $\psi_{A_i}^{s_i} \equiv \phi_{A_i}^{s_i\,\mathrm{T}}$, where the transpose is taken in the computational basis.

$$\chi_{A_1\cdots A_n} = \sum_{s_1,\cdots,s_n} \beta_{s_1\cdots s_n} \bigotimes_{i=1}^{n} (\psi_{A_i}^{s_i})^{\mathrm{T}} \tag{D.4}$$

In the semiquantum game, the referee gives one of the pure states $\psi_{A_i^o}^{s_i} \in \mathscr{B}_{A_i^o}^{proj\ \mathrm{T}}$ to the i^{th} party in each run (See the proof of Theorem 2).

Appendix E

Necessity of non-orthogonal inputs in Theorem 8 and Theorem 9

In this section, we will discuss the necessity of the non-orthogonal quantum inputs in the game $\mathbb{G}_{sq}$ used in Theorem **1** and Theorem **2**. While paying the game $\mathbb{G}_{sq}$, let the i^{th} party get the quantum input $\psi^{s_i}_{A^o_i} \in \mathscr{D}(\mathscr{H}_{A^o_i})$ and perform some joint measurement $M_{A^o_i A_i} \equiv \{\pi^{a_i}_{A^o_i A_i}\}$, where a_i is the outcome corresponding to the POVM element $\pi^{a_i}_{A^o_i A_i}$. For the BQS $W_{A_1 \cdots A_n}$, the joint probabilities are given by

$$
\begin{aligned}
p(a_1, \cdots, a_n | \psi^{s_1}_{A^o_1}, \cdots, \psi^{s_n}_{A^o_n}) &= \mathrm{Tr}\left[\bigotimes_i \pi^{a_i}_{A^o_i A_i}\left(\bigotimes_i \psi^{s_i}_{A^o_i} \otimes W_{A_1 \cdots A_n}\right)\right] \\
&\equiv \mathrm{Tr}\left[\left(\bigotimes_i Q^{a_i}_{A_i}[s_i]\right) W_{A_1 \cdots A_n}\right], \qquad \text{(E.1)}
\end{aligned}
$$

where, $Q^{a_i}_{A_i}[s_i] := \mathrm{Tr}_{A^o_i}\left[\pi^{a_i}_{A_i A^o_i}(I_{A_i} \otimes \psi^{s_i}_{A^o_i})\right] \in \mathscr{E}\left(\mathscr{H}_{A_i}\right)$ effectively acts on A_i subsystem of the shared state $W_{A_1 \cdots A_n}$ when the quantum input $\psi^{s_i}_{A^o_i}$ is given by the referee. Since $\sum_{a_i} \pi^{a_i}_{A_i A^o_i} = \mathbb{I}_{A_i A^o_i}$, we have,

$$
\sum_{a_i} Q^{a_i}_{A_i}[s_i] = \mathrm{Tr}_{A^o_i}\left[\left(\sum_{a_i} \pi^{a_i}_{A_i A^o_i}\right)\left(\mathbb{I}_{A_i} \otimes \psi^{s_i}_{A^o_i}\right)\right] = \mathrm{Tr}_{A^o_i}\left[\mathbb{I}_{A_i} \otimes \psi^{s_i}_{A^o_i}\right] = \mathbb{I}_{A_i}. \qquad \text{(E.2)}
$$

Therefore, $M^{s_i}_{A_i} \equiv \left\{Q^{a_i}_{A_i}[s_i]\right\}$ is the effective measurement performed by the A^{th}_i party on the shared state $W_{A_1 \cdots A_n}$ when the quantum input $\psi^{s_i}_{A^o_i}$ is received by the i^{th} party.

Let us now assume that the BQS is of the following form

$$W_{A_1\cdots A_n} = \sum_k p_k \left(\Lambda^k_{A_1} \otimes \cdots \otimes \Lambda^k_{A_n}\right) \rho^k \tag{E.3}$$

where, $\rho^k \in \mathscr{D}(\otimes_i \mathscr{H}_{A_i})$, Λ^k_i are positive trace-preserving maps, and $\{p_k\}$ is a probability distribution. In this case we have,

$$\begin{aligned} p\left(a_1, \cdots, a_n | \psi^{s_1}_{A^o_1}, \cdots, \psi^{s_n}_{A^o_n}\right) &= \mathrm{Tr}\left[\left(\bigotimes_i Q^{a_i}_{A_i}[s_i]\right) \sum_k p_k \left(\bigotimes_i \Lambda^k_{A_i}\right) \rho^k\right] \\ &= \sum_k p_k \,\mathrm{Tr}\left[\left(\bigotimes_i Q^{a_i}_{A_i}[s_i]\right)\left(\bigotimes_i \Lambda^k_{A_i}\right) \rho^k\right] \\ &= \sum_k p_k \,\mathrm{Tr}\left[\left\{\bigotimes_i \Lambda^{*k}_{A_i}\left(Q^{a_i}_{A_i}[s_i]\right)\right\} \rho^k\right] \\ &= \sum_k p_k \,\mathrm{Tr}\left[\left(\bigotimes_i \tilde{Q}^{a_i}_{A_i}[s_i]\right) \rho^k\right], \end{aligned} \tag{E.4}$$

where Λ^* is the adjoint map of Λ, *i.e.* $\mathrm{Tr}[U\Lambda(V)] = \mathrm{Tr}[\Lambda^*(U)V]$ for all Hermitial matrices U and V. Clearly $\tilde{M}^{s_i}_{A_i} \equiv \{\tilde{Q}^{a_i}_{A_i}[s_i]\}$ is a valid quantum measurement since the dual of a positive trace-preserving map is positive and unital. Therefore, for the class of BQSs given by Eq.(E.3), whenever the input states are orthogonal, the correlations generated by the BQS can by simulated quantum mechanically as follows:

The i^{th} party first performs a measurement to identify the index 's_i' of the given quantum state $\psi^{s_i}_{A^o_i}$ and then performs the measurement $\tilde{M}^{s_i}_{A_i} \equiv \{\tilde{Q}^{a_i}_{A_i}[s_i]\}$ on her part of the multipartite quantum state ρ^k. This generates the correlation $p(a_1, \cdots, a_n | \psi^{s_1}_{A^o_1}, \cdots, \psi^{s_n}_{A^o_n})$ which was obtained by performing the local measurements $M_{A^o_i A_i} \equiv \{\pi^{a_i}_{A^o_i A_i}\}$ on $\bigotimes_i \psi^{s_i}_{A^o_i} \otimes W_{A_1\cdots A_n}$. Note that if the inputs are orthogonal then the index 's_i' can be identified unambiguously. Therefore, when the BQSs are of the form (E.3), non-orthogonal inputs are necessary to obtain the advantage of BQS over quantum states. This reproduces the results in [115, 130].